# A REMARK ON VARIETIES OF LATTICES AND SEMIGROUPS[1]

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We consider for any variety $V$, the lattice $L(V)$ of subvarieties of $V$. It is known (H. Neumann [2]) that if $V$ is the variety of groups, then $V$ is not the join of any finite number of proper subvarieties. Recently, B. Jónsson [1] has proved a similar result for the variety of lattices. His proof though short requires rather sophisticated machinery. In this remark we show by elementary methods that for any two lattice identities there is a nontrivial identity which each implies independently. A similar approach also gives the corresponding result for semigroups. A trivial observation used in the lemmas is that $h_1=h_2$ is a nontrivial identity if and only if $h_1$ and $h_2$ represent distinct elements of a free algebra. From these two lemmas it follows easily that for neither lattices nor semigroups is the variety a join of a finite number of subvarieties.

**1. Lattices.** We begin with the easily verified remark that if $e_1(x_1, x_2, \cdots)=e_2(x_1, x_2, \cdots)$ is a nontrivial lattice identity (i.e. one not satisfied in all lattices), then there is a nontrivial identity $f_1=f_2$ implied by $e_1=e_2$ such that $f_1 \leqq f_2$ is trivial. Indeed, $f_1=e_1 \cap e_2$, $f_2 = e_1 \cup e_2$ suffices.

LEMMA 1. *Let $f_1=f_2$ and $g_1=g_2$ be nontrivial lattice identities. There exists a nontrivial lattice identity $h_1=h_2$ which is implied by each of $f_1=f_2$ and $g_1=g_2$.*

PROOF. By our initial remark it is sufficient to prove this in the case where $f_1 \leqq f_2$ and $g_1 \leqq g_2$ are trivial. Let $f_1=f_2$ involve $m$ variables and $g_1=g_2$ involve $n$ variables. Consider the free lattice $F$ on the $m+n$ generators, $x_1, \cdots, x_m, y_1, \cdots, y_n$. Let $X=\langle x_1, \cdots, x_m\rangle$ and $Y=\langle y_1, \cdots, y_n\rangle$. From our assumption, we have that in $F$, $f_1(X) \leqq f_2(X)$ and $g_1(Y) \leqq g_2(Y)$.

We take $h_1=h_2$ to be the identity

$$f_1(X) \cup g_1(Y) = f_1(X) \cup g_1(Y) \cup (f_2(X) \cap g_2(Y))$$

where now, of course, $X$ and $Y$ denote variables. Obviously $h_1 \leqq h_2$ is satisfied in all lattices. It is also easy to see that the identity $h_1=h_2$

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is implied by each of the identities $f_1=f_2$ and $g_1=g_2$. It remains to show that $h_1=h_2$ is not trivial.

If $h_1 \geqq h_2$ holds in $F$, then also holding in $F$ is

$$f_1(X) \cup g_1(Y) \geqq f_2(X) \cap g_2(Y).$$

From Whitman's description [3] this relation can hold only if one of four possibilities holds. One of these is $f_1(X) \geqq f_2(X) \cap g_2(Y)$. However, if this holds in $F$, the endomorphism $F \rightarrow F$ which sends $x_i \rightarrow x_i$ and $y_j \rightarrow \bigcup X$ for all $x_i \in X$ and $y_j \in Y$ implies that $f_1(X) \geqq f_2(X)$ in $F$, a contradiction. The other three possibilities are the natural symmetric and dual variations of this one and cannot hold here without a contradiction.

2. **Semigroups.** We begin with a technical lemma for later convenience.

LEMMA 2. *Let $e_1(x_1, \cdots, x_m)=e_2(x_1, \cdots, x_m)$ be a nontrivial semigroup identity. There exists a nontrivial identity $f_1(x_1, x_2)=f_2(x_1, x_2)$ in two variables implied by $e_1=e_2$ such that $f_1$ and $f_2$ are words in $x_1$ and $x_2$ both of the same length.*

PROOF. Consider the free semigroup $F$ on the $2m$ generators $x_1, \cdots, x_m, y_1, \cdots, y_m$. Let $X=\langle x_1, \cdots, x_m\rangle$ and $Y=\langle y_1, \cdots, y_m\rangle$. From our assumption $e_1(X) \neq e_2(X)$ in $F$, and thus $e_1(X)e_2(Y) \neq e_2(X)e_1(Y)$ in $F$. The two sides are of the same length, and for some $n$, the $n$th place in $e_1(X)e_2(Y)$ and the $n$th place in $e_2(X)e_1(Y)$ are occupied by different variables, say $z_1$, $z_2$ respectively. In $e_1(X)e_2(Y)$ and $e_2(X)e_1(Y)$ put all variables but $z_1$ equal to $z_2$. Equating the expressions so obtained yields an identity in $z_1$, $z_2$ which is a consequence of $e_1(X)=e_2(X)$.

LEMMA 3. *If $f_1=f_2$ and $g_1=g_2$ are nontrivial, then there is a nontrivial identity $h_1=h_2$ which is implied by each.*

PROOF. By Lemma 2 we may, without loss of generality, assume that the two identities of the hypothesis are each in two variables $x$, $y$ and that $f_1, f_2$ and $g_1$, $g_2$ have the same length. For $h_1=h_2$ we take

$$\begin{aligned} &f_1(g_1(x, y), g_2(x, y))f_2(g_1(x, y), g_2(x, y)) \\ &\qquad = f_2(g_1(x, y), g_2(x, y))f_1(g_1(x, y), g_2(x, y)). \end{aligned}$$

Clearly $h_1=h_2$ is implied by either one of the original identities. We claim that $h_1=h_2$ is nontrivial. To begin with $f_1(x, y)$ and $f_2(x, y)$ must differ at some place, say the $n$th, and $g_1(x, y)$ and $g_2(x, y)$ must differ at some place, say the $m$th. Thus $h_1$, as a word in $g_1$, $g_2$, must have, say $g_1$, at the $n$th place, while $h_2$ has $g_2$ at this place. But now in the substitution for $g_1$ and $g_2$ by their respective words on $x$ and $y$, the $m$th place of these components will differ in the symbols $x$ and $y$. Thus $h_1=h_2$ is nontrivial.

## 3. The lattice of varieties.

THEOREM. *Let $V$ be either the variety of all lattices or the variety of all semigroups. In $L(V)$, $V$ is not the join of a finite number of proper subvarieties.*

PROOF. By an obvious induction it suffices to show that the join of two proper subvarieties is a proper subvariety.

If $V_1$ and $V_2$ are two distinct proper subvarieties with neither contained in the other, there is an identity $f_1=f_2$ satisfied by $V_1$ but not by $V_2$, and conversely an identity $g_1=g_2$ satisfied by $V_2$ but not by $V_1$.

If $V_1'$ is the variety defined by $f_1=f_2$ and $V_2'$ is the variety defined by $g_1=g_2$, then $V_1\leqq V_1'$ and $V_2\leqq V_2'$. Furthermore, with $h_1=h_2$ as in Lemma 1 or Lemma 3, the variety $W$ defined by $h_1=h_2$ contains $V_1'$ and $V_2'$, hence contains $V_1\cup V_2$. But by the lemma, $W$ is a proper subvariety. Thus $V_1\cup V_2$ is a proper subvariety of $V$.

## REFERENCES

1. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. (to appear).

2. H. Neumann, *Varieties of groups*, Springer-Verlag, Berlin, 1967.

3. P. M. Whitman, *Free lattices*. I, Ann. of Math. (2) **42** (1941), 325–330.

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