A CLASS OF GROUPS WHOSE LOCAL SEQUENCE IS NONSTATIONARY

KENNETH K. HICKIN[1]

Let Σ be a class of groups. Define the local operator L as follows:

(i) $L^0(\Sigma)=\Sigma$.

(ii) If $\alpha>0$ is an ordinal number, then $L^\alpha(\Sigma)=$ the class of all groups having an upper-directed cover of subgroups, each belonging to the class $\bigcup\{L^\beta(\Sigma)\mid\beta<\alpha\}$.

We will consider all classes of groups to be isomorphism-closed. $L^1(\Sigma)$ is the local system defined in [1, p. 166]. It is well known that if Σ is closed under the taking of subgroups then $L^2(\Sigma)=L^1(\Sigma)$.

In the following, for each ordinal α of cardinality $\leqq c$, the continuum, a class of groups will be displayed whose local sequence does not become stationary before α iterations.

First define an equivalent operator for sets: Let Γ be a set of sets. Define the operator C as follows:

(i) $C^0(\Gamma)=\Gamma$.

(ii) If $\alpha>0$ is an ordinal number, then $C^\alpha(\Gamma)=$ the set of all sets having an upper-directed cover of subsets, each belonging to the set $\bigcup\{C^\beta(\Gamma)\mid\beta<\alpha\}$.

If S is a set, denote its power set by $P(S)$; if Γ is a set of sets, we will sometimes call $\bigcup\Gamma$ the "underlying set" of Γ.

For any set of sets Γ and ordinal α, we have that $C^\alpha(\Gamma)\subset P(\bigcup\Gamma)$. Thus all such set-theoretic sequences must eventually become stationary, and we may define $|\Gamma|$ to be the smallest ordinal such that $C^{|\Gamma|+1}(\Gamma)=C^{|\Gamma|}(\Gamma)$.

We wish first to solve the set-theoretic problem by displaying, for any ordinal α, a set of sets Γ satisfying $|\Gamma|=\alpha$ (Lemma 1). After the following definitions, a proposition to be used in Lemma 1 will be proved.

DEFINITION. Suppose for each $\alpha\in A$, Γ_α is a set of sets. Define $\Sigma\{\Gamma_\alpha\mid\alpha\in A\}=\{\bigcup f(A)\mid$ where $f\colon A\to\bigcup\{\Gamma_\alpha\mid\alpha\in A\}$ is a function such that $\forall\alpha\in A$, $f(\alpha)\in\Gamma_\alpha\}$. That is, an element of $\Sigma\{\Gamma_\alpha\mid\alpha\in A\}$ is a union of sets, one chosen from each Γ_α.

DEFINITION. Suppose Γ is a set of sets and $S\in C^\alpha(\Gamma)$ for some ordinal α. Thus S possesses an upper-directed cover $\{X_\mu\mid\mu\in M\}$ of sub-

Received by the editors April 24, 1968.

[1] The research for this paper was sponsored by National Science Foundation Grant number GY-2880.

sets such that each $X_\mu \in C^\beta(\Gamma)$ for some $\beta<\alpha$. We will say that the cover $\{X_\mu \mid \mu \in M\}$ is "augmented" if $\exists \mu \in M$ such that $X_\mu \in \Gamma$.

It is easy to see that any $S \in C^\alpha(\Gamma)$, for any α, possesses an augmented cover.

PROPOSITION. *Suppose for each $\alpha \in A$, Γ_α is a set of sets such that the collection $\{\cup\Gamma_\alpha \mid \alpha \in A\}$ of underlying sets is pairwise disjoint. Let $\Gamma = \Sigma\{\Gamma_\alpha \mid \alpha \in A\}$. Then* (i) *for any ordinal β, $C^\beta(\Gamma) = \Sigma\{C^\beta(\Gamma_\alpha) \mid \alpha \in A\}$, and* (ii) $|\Gamma| = \sup\{|\Gamma_\alpha| \mid \alpha \in A\}$.

PROOF. PROOF OF (i). If $\beta=0$, the assertion is immediate. Suppose $\beta>0$ and $\forall \rho<\beta$, $C^\rho(\Gamma) = \Sigma\{C^\rho(\Gamma_\alpha) \mid \alpha \in A\}$.

Assume $B \in \Sigma\{C^\beta(\Gamma_\alpha) \mid \alpha \in A\}$, so that $B = \cup f(A)$, where f: $A \to \cup\{C^\beta(\Gamma_\alpha) \mid \alpha \in A\}$ is a function such that $\forall \alpha \in A$, $f(\alpha) \in C^b(\Gamma_\alpha)$. Thus each $f(\alpha) = B_\alpha$ has an upper-directed, augmented cover $\{X_\alpha^\psi \mid \psi \in I_\alpha\}$ of subsets X_α^ψ, where each X_α^ψ is a member of $C^\rho(\Gamma_\alpha)$ for some $\rho<\beta$. Since the covers are augmented, for each α let $X_\alpha^{\psi_0} \in \Gamma_\alpha$. Let $Y_0 = \cup\{X_\alpha^{\psi_0} \mid \alpha \in A\}$; thus $Y_0 \in \Gamma = \Sigma\{\Gamma_\alpha \mid \alpha \in A\}$. For each $\rho<\beta$ define $\Lambda_\rho \subset \Sigma\{C^\rho(\Gamma_\alpha) \mid \alpha \in A\} = C^\rho(\Gamma)$ as follows:

$$\begin{aligned} Y \in \Lambda_\rho \Leftrightarrow Y &= \cup\{X_\alpha^{\psi(\alpha)} \mid \alpha \in A, \text{ where } X_\alpha^{\psi(\alpha)} \in C^\rho(\Gamma_\alpha) \\ &\qquad \text{and for all but a finite number of } \alpha,\ X_\alpha^{\psi(\alpha)} = X_\alpha^{\psi_0}\}. \end{aligned}$$

Thus if we put $\Lambda = \cup\{\Lambda_\rho \mid \rho<\beta\} \subset \cup\{C^\rho(\Gamma) \mid \rho<\beta\}$, in order to show $B \in C^\beta(\Gamma)$, it suffices to show that Λ is an upper-directed cover for B.

To show that Λ covers B, select any $b \in B$. Then $\exists \alpha', \psi'$ such that $b \in X_{\alpha'}^{\psi'}$, since $\{X_\alpha^\psi \mid \alpha \in A,\ \psi \in I_\alpha\}$ covers B. Also $\exists \rho<\beta$ such that $X_{\alpha'}^{\psi'} \in C^\rho(\Gamma_{\alpha'})$. Thus if we let

$$Y = \cup\{X_\alpha^\psi \mid \alpha \in A, \text{ where } X_\alpha^\psi = X_{\alpha'}^{\psi'} \text{ when } \alpha = \alpha', \text{ but } X_\alpha^\psi = X_\alpha^{\psi_0} \text{ when } \alpha \neq \alpha'\},$$

then $Y \in \Lambda_\rho \subset \Lambda$ and $b \in Y$. Thus Λ covers B.

To show that Λ is upper-directed, let Y_1, $Y_2 \in \Lambda$. Say $Y_1 = \cup\{X_\alpha^{\phi_\alpha} \mid \alpha \in A\}$ and $Y_2 = \cup\{X_\alpha^{\theta_\alpha} \mid \alpha \in A\}$. Let $\alpha_1, \cdots, \alpha_n$ be elements of A such that $\forall \alpha \in A$, $\alpha \notin \{\alpha_1, \cdots, \alpha_n\} \Rightarrow X_\alpha^{\theta_\alpha} = X_\alpha^{\psi_0}$ and $X_\alpha^{\phi_\alpha} = X_\alpha^{\psi_0}$. Such a set $\{\alpha_1, \cdots, \alpha_n\}$ exists by the finiteness condition in the definition of Λ_ρ. Since $\forall \alpha \in A$, $\{X_\alpha^\psi \mid \psi \in I_\alpha\}$ is an upper-directed cover of B_α, we have that for each α_i, $1 \leqq i \leqq n$, $\exists \psi_i \in I_{\alpha_i}$ such that $X_{\alpha_i}^{\theta_{\alpha_i}} \subset X_{\alpha_i}^{\psi_i}$ and $X^{\phi_{\alpha_i}} \subset X_{\alpha_i}^{\psi_i}$. Also $\exists \rho_i<\beta$ such that $X_{\alpha_i}^{\psi_i} \in C^{\rho_i}(\Gamma)_{\alpha_i}$. Letting $\rho' = \max\{\rho_i \mid 1 \leqq i \leqq n\} < \beta$, we have $Y = \cup\{X_{\alpha_i}^{\psi_i} \mid 1 \leqq i \leqq n\} \cup (\cup\{X_\alpha^{\psi_0} \mid \alpha \notin \{\alpha_1, \cdots, \alpha_n\},\ \alpha \in A\}) \in \Lambda_{\rho'} \subset \Lambda$, and $Y_1 \subset Y$, $Y_2 \subset Y$. Thus Λ is upper-directed.

It follows that $B \in C^\beta(\Gamma)$, and hence $\Sigma\{C^\beta(\Gamma_\alpha) \mid \alpha \in A\} \subset C^\beta(\Gamma)$.

Now assume $B \in C^{\beta}(\Gamma)$. Then B possesses an upper-directed cover $\{Y_\mu \mid \mu \in M\}$ where each Y_μ is an element of $C^{\rho_\mu}(\Gamma)$ for some $\rho_\mu < \beta$. Thus $Y_\mu \in \Sigma\{C^{\rho_\mu}(\Gamma_\alpha) \mid \alpha \in A\}$, and we may write

$$Y_\mu = \cup\{X_\alpha^\mu \mid \alpha \in A, \text{ where } X_\alpha^\mu \in C^{\rho_\mu}(\Gamma_\alpha)\}.$$

By the disjointness property of the sets underlying the Γ_α, we have that $B_\alpha = B \cap (\cup\Gamma_\alpha) = \cup\{X_\mu^\alpha \mid \mu \in M\}$ and, since $\{Y_\mu \mid \mu \in M\}$ is upper-directed, $\{X_\alpha^\mu \mid \mu \in M\}$ is also. Thus $\forall \alpha \in A$, $B_\alpha \in C^{\beta}(\Gamma_\alpha)$ and $B = \cup\{B_\alpha \mid \alpha \in A\}$. It follows that $B \in \Sigma\{C^{\beta}(\Gamma_\alpha) \mid \alpha \in A\}$, and hence $C^{\beta}(\Gamma) \subset \Sigma\{C^{\beta}(\Gamma_\alpha) \mid \alpha \in A\}$.

The proof of (i) is now complete.

Proof of (ii). This follows immediately from (i) and the disjointness of the sets underlying the Γ_α. For if $\beta < |\Gamma_\alpha|$ for some α, then $C^{\beta+1}(\Gamma_\alpha) > C^{\beta}(\Gamma_\alpha) \Rightarrow C^{\beta+1}(\Gamma) = \Sigma\{C^{\beta+1}(\Gamma_\alpha) \mid \alpha \in A\} > \Sigma\{C^{\beta}(\Gamma_\alpha) \mid \alpha \in A\} = C^{\beta}(\Gamma)$, which shows $|\Gamma| \geqq \sup\{|\Gamma_\alpha| \mid \alpha \in A\} = \sigma$. On the other hand, $C^{\sigma+1}(\Gamma) = \Sigma\{C^{\sigma+1}(\Gamma_\alpha) \mid \alpha \in A\} = \Sigma\{C^{\sigma}(\Gamma_\alpha) \mid \alpha \in A\} = C^{\sigma}(\Gamma)$, which shows $|\Gamma| \leqq \sigma$.

Lemma 1. *For all ordinals α there exists a set of sets Γ_α satisfying $|\Gamma_\alpha| = \alpha$. If α is infinite, of cardinality $\bar{\alpha}$, then Γ_α can be chosen so that $\cup\Gamma_\alpha$ has cardinality $\bar{\alpha}$.*

Proof. We induct on the theorem and on the additional property $\cup\Gamma_\alpha \in C^{\alpha}(\Gamma_\alpha)$, but $\forall \beta < \alpha$, $\cup\Gamma_\alpha \notin C^{\beta}(\Gamma_\alpha)$. The theorem follows when $\alpha = 0$ trivially and when $\alpha = 1$, letting

$$\Gamma_1 = \{\{1, 2, \cdots, n\} \mid n \in N, \text{ the natural numbers}\},$$

we have that $N \notin \Gamma_1$, but $N \in C^1(\Gamma_1) = C^2(\Gamma_1)$.

Assume the theorem and the additional property hold for all ordinals less than α, $\alpha > 1$.

Case 1. α is a limit ordinal. For each $\beta < \alpha$ choose Γ_β satisfying the inductive hypotheses such that the collection $\{\cup\Gamma_\beta \mid \beta < \alpha\}$ of underlying sets is pairwise disjoint. Define $\Gamma_\alpha = \Sigma\{\Gamma_\beta \mid \beta < \alpha\}$. By (ii) of the proposition we have immediately that $|\Gamma_\alpha| = \alpha$. Now $\cup\Gamma_\alpha = \cup\{\cup\Gamma_\beta \mid \beta < \alpha\}$, and each $\cup\Gamma_\beta \in C^{\beta}(\Gamma_\beta)$, implying by (i) of the proposition $\cup\Gamma_\alpha \in \Sigma\{C^{\alpha}(\Gamma_\beta) \mid \beta < \alpha\} = C^{\alpha}(\Gamma_\alpha)$. On the other hand, if $\mu < \alpha$, then $\cup\Gamma_{\mu+1} \notin C^{\mu}(\Gamma_{\mu+1})$, and hence $\cup\Gamma_\alpha \notin \Sigma\{C^{\mu}(\Gamma_\beta) \mid \beta < \alpha\} = C^{\mu}(\Gamma)$. Thus Γ_α satisfies all inductive hypotheses.

Case 2. $\alpha = \gamma + 1$. Let Λ be a set of sets satisfying $|\Lambda| = \gamma$ and the other inductive hypotheses. Let $\{\Lambda_i \mid i = 1, 2, \cdots\}$ be copies of Λ obtained by indexing the elements of $\cup\Lambda$ with the i's so that the collection $\{\cup\Lambda_i \mid i = 1, 2, \cdots\}$ of underlying sets is pairwise disjoint. Define

$$\Gamma_\alpha = \{X \mid X = L_1 \cup L_2 \cup \cdots \cup L_{N-1} \cup (\cup\Lambda_N), \text{ where } L_i \in \Lambda_i, N \geqq 1\}.$$

Suppose for all μ such that $\mu<\beta\leqq\gamma$, any $X\in C^{\mu}(\Gamma)$ is of the form $S_1\cup\cdots\cup S_{N-1}\cup(\cup\Lambda_N)$ where $\forall i=1,\cdots,N-1$, $S_i\in C^{\mu}(\Lambda_i)$. Call N the length of X. (This is clearly so if $\beta=1$.) Let $\Delta=\cup\{C^{\mu}(\Gamma_\alpha)\mid\mu<\beta\}$. Suppose some member X of $C^{\beta}(\Gamma_\alpha)$ is realized by the upper-directed cover $\{X_\rho\mid\rho\in R\}$ where $\forall\rho\in R$, $X_\rho\in\Delta$. Further, let $\rho_1,\rho_2\in R$ be such that $X_{\rho_1}\subset X_{\rho_2}$. Say $X_{\rho_1}=S_1\cup\cdots\cup S_{N-1}\cup(\cup\Lambda_N)$, $S_i\in C^{\mu_1}(\Lambda_i)$, $\mu_1<\beta$, and $X_{\rho_2}=T_1\cup\cdots\cup T_{M-1}\cup(\cup\Lambda_M)$, $T_i\in C^{\mu_2}(\Lambda_i)$, $\mu_2<\beta$. If $N\neq M$, it must be the case that either $S_i=\cup\Lambda_i$ for some i, or $T_i=\cup\Lambda_i$ for some i, which is impossible by the inductive hypotheses on $\cup\Lambda$. Thus $N=M$. Hence all members of the cover $\{X_\rho\mid\rho\in R\}$ must have the same length N, and we may write $X_\rho=S_1^\rho\cup\cdots\cup S_{N-1}^\rho\cup(\cup\Lambda_N)$, where $S_i^\rho\in C^{\mu}(\Lambda_i)$ for some $\mu<\beta$. The disjointness of the underlying sets of the Λ_i now yields that, for $1\leqq i\leqq N-1$, $\{S_i^\rho\mid\rho\in R\}$ is an upper-directed cover for $X\cap(\cup\Lambda_i)$. Thus X, an arbitrary element of $C^{\beta}(\Lambda_\alpha)$, is of the form $T_1\cup\cdots\cup T_{N-1}\cup(\cup\Lambda_N)$, where $T_i\in C^{\beta}(\Lambda_i)$, $1\leqq i\leqq N-1$.

In particular, the above argument shows that the members of $C^{\gamma}(\Gamma_\alpha)$ are of the form

$$(*)\qquad S_1\cup\cdots\cup S_{N-1}\cup(\cup\Lambda_N),\qquad S_i\in C^{\gamma}(\Lambda_i),\quad 1\leqq i\leqq N-1.$$

Since $|\Lambda|=\gamma$ and $\cup\Lambda\in C^{\gamma}(\Lambda)$, it follows that for each $N\geqq 1$, $(\cup\Lambda_1)\cup\cdots\cup(\cup\Lambda_N)\in C^{\gamma}(\Gamma_\alpha)$. Hence $\cup\Gamma_\alpha=\cup\{\cup\Lambda_i\mid i=1,2,\cdots\}$ has an upper-directed cover of subsets in $C^{\gamma}(\Gamma_\alpha)$. But $\cup\Gamma_\alpha\notin C^{\gamma}(\Gamma_\alpha)$ since $\cup\Gamma_\alpha$ is not of the form (*).

It remains to show that $C^{\gamma+1}(\Gamma_\alpha)=C^{\gamma+2}(\Gamma_\alpha)$. We will omit the details; however, from the form (*), the following characterization of the members of $C^{\gamma+1}(\Gamma_\alpha)$ is easily obtained: $X\in C^{\gamma+1}(\Gamma_\alpha)\Leftrightarrow X=S_1\cup\cdots\cup S_N\cup\cdots$ where either (1) each $S_i\in C^{\gamma}(\Lambda_i)$ and $S_i=\cup\Lambda_i$ cofinally in $S_1,\cdots,S_N,\cdots$ or (2) $X\in C^{\gamma}(\Gamma_\alpha)$ (and hence the S_i are empty after a point).

From this it is easy to see that any directed system of sets in $C^{\gamma+1}(\Gamma_\alpha)$ again yields a member of $C^{\gamma+1}(\Gamma_\alpha)$.

Thus Γ_α satisfies the inductive hypotheses.

The example given for $|\Gamma_1|=1$ at the outset was such that $\cup\Gamma_1$ had cardinality d of the natural numbers. If α is a nonlimit ordinal, $\alpha=\gamma+1$, and $\cup\Gamma_\gamma$ is of infinite cardinality σ, then Γ_α, as constructed, has cardinality $d\sigma=\sigma$. Thus, as constructed, $\cup\Gamma_\omega$ has cardinality $d=\bar{\omega}$, since at limit ordinals α, $\cup\Gamma_\alpha$ will have cardinality $\sum_{\beta<\alpha}\sigma_\beta$ where σ_β is the cardinality of $\cup\Gamma_\beta$.

It is thus clear that for all infinite ordinals α, $\cup\Gamma_\alpha$ will have cardinality $\bar{\alpha}$, and the proof is complete.

LEMMA 2. *If A is a countably infinite set then there exists an uncountable set of subsets of A such that no containments hold between distinct members.*

PROOF. Let $\{A_i \mid i=1, 2, \cdots\}$ be a partition of A such that each A_i is countably infinite. Define $K \subset P(A)$ by $K=\{B \subset A \mid B$ contains exactly one element from each $A_i\}$. The cardinality of K is $d^d=2^d$, and if $B_1, B_2 \in K$, clearly $B_1 \not\subset B_2$, unless $B_1=B_2$.

COROLLARY. *There exists a set of cardinality c of torsion abelian groups, $T=\{T_\alpha \mid \alpha \in K\}$ such that $\forall \alpha_1, \alpha_2 \in K, \alpha_1 \neq \alpha_2 \Rightarrow T_{\alpha_1} \not\subsetneq T_{\alpha_2}$.*

PROOF. Let A be a countably infinite set of primes. Applying Lemma 2, $\exists K \subset P(A)$ such that K is uncountable and for any B_1, $B_2 \in K$, $B_1 \neq B_2 \Rightarrow B_1$ contains some prime not in B_2. For $B \in K$, define $T_B=\sum_{p \in B} J_p$ (the direct sum), where J_p is a cyclic group of order p. We claim that the set of groups $\{T_B \mid B \in K\}$ is the desired set. For suppose $B_1, B_2 \in K$ and $B_1 \neq B_2$. Then if p is a prime in $B_1 \backslash B_2$, we have that T_{B_1} has an element of order p, whereas T_{B_2} does not. Hence $T_{B_1} \not\subsetneq T_{B_2}$.

We will also need several properties of free products of groups.

LEMMA 3. *Let $\{G_\delta \mid \delta \in \Delta\}$ and $\{H_\lambda \mid \lambda \in \Lambda\}$ be arbitrary collections of groups, and put $R=(\prod^*_{\delta \in \Delta} G_\delta)*(\prod^*_{\lambda \in \Lambda} H_\lambda)$. Then $(\prod^*_{\delta \in \Delta} G_\delta) \cap \langle H^R_\gamma \mid \gamma \in \Lambda \rangle$ is trivial.*

LEMMA 4. *If $\{G_\delta \mid \delta \in \Delta\}$ is a collection of groups, and H is a freely indecomposable group, but not infinite cyclic, satisfying $H \subset G = \prod^*_{\delta \in \Delta} G_\delta$, then $\exists \delta \in \Delta$ such that H is a subgroup of a conjugate of G_δ in G.*

PROOF. Suppose $H \subset G=\prod^*_{\delta \in \Delta} G$ where H is freely indecomposable but not infinite cyclic. By the subgroup theorem for free products [1, p. 17], $H=F*\prod^*_{v \in V} H_v$ where F is a free group and $\forall v \in V$, H_v is conjugate in G to a subgroup of G_δ for some $\delta \in \Delta$. By another theorem [1, p. 26], any two free decompositions of a group possess isomorphic refinements. Hence, since H is freely indecomposable, $F*\prod^*_{v \in V} H_v$ must have exactly one nontrivial factor. H cannot be isomorphic to F since the only freely indecomposable free group is infinite cyclic (or trivial). Thus $H=H_v$ for some $v \in V$. The lemma follows.

We can now prove the desired theorem.

THEOREM. *For any ordinal α of cardinality $\leqq c$, there exists a class of groups B_α such that $L^\alpha(B_\alpha)=L^{\alpha+1}(B_\alpha)$, but $L^\beta(B_\alpha)<L^{\beta+1}(B_\alpha)$ when $\beta<\alpha$.*

PROOF. Let $T=\{T_\alpha|\alpha\in K\}$ be the class of torsion abelian groups of the corollary to Lemma 2. By virtue of Lemma 1, $\exists P\subset P(K)$ such that $|P|=\alpha$. If $Y\subset K$, define $F_Y=\prod^*_{\gamma\in Y} T_\gamma$ and put $F=\{F_Y|\ Y\subset K\}$.

Define the class $B_\alpha=\{F_Y|\ Y\in P\}$.

Suppose $\{F_l|\,l\in\Lambda\}$ is an upper-directed cover of subgroups in F for some $G\in F$, $G=F_Y$. We assert that the set of sets $\{l|\,l\in\Lambda\}$ is an upper-directed cover of subsets for Y, since

(1) If $F_{l_1}\subset F_{l_2}$, then each free factor of F_{l_1}, by Lemma 4, is isomorphically contained in some free factor of F_{l_2}, and so by the property of the $\{T_\alpha|\alpha\in K\}$ these free factors are isomorphic. This shows $l_1\subset l_2$. Hence $\{l|\,l\in\Lambda\}$ is upper-directed provided $\{F_l|\,l\in\Lambda\}$ is also.

(2) Consider, by Lemma 4, all of the conjugate subgroups of free factors of G to which the free factors of the $\{F_l|\,l\in\Lambda\}$ belong. If no conjugate of some free factors of G occurs, then Lemma 3 is violated since $\{F_l|\,l\in\Lambda\}$ covers G. Hence all free factors of G are represented, and so $\{l|\,l\in\Lambda\}$ covers Y.

Thus any such group-theoretic covering yields a set-theoretic covering according to the correspondence $T_x\to x$. Likewise any set-theoretic covering yields a group-theoretic covering.

Considering the local sequence B_α, $L(B_\alpha)$, $\cdots$ $L^\beta(B_\alpha)$, $\cdots$ the theorem will be proved if we can eliminate the possibility that, at some stage in the sequence of local covers leading to any $G\in L^\beta(B_\alpha)\cap F$, some group $H\notin F$ occurs. Since the sequence of local covers leading to G is well-ordered, such an H must occur for a first time at some stage. Hence, WLOG, we may assume that H possesses an upper-directed cover $\{F_\gamma|\gamma\in\Gamma\}$ of subgroups in F.

Since $H\subset G\in F$, by the subgroup theorem for free products we have $H\approx Q*\prod^*_{\rho\in R}\tau_\rho$ where Q is a free group and each τ_ρ is isomorphic to a subgroup of some member of T. Each free factor T_{γ_i} of each F_γ, $\gamma\in\Gamma$, by Lemma 4, is isomorphically contained in some τ_ρ, and hence, for each such τ_ρ, $\tau_\rho\approx T_{\alpha_\rho}$. Since $\{F_\gamma|\gamma\in\Gamma\}$ covers H, by Lemma 3, all of the τ_ρ, $\rho\in R$, are obtained in this way, and consequently $H\approx\prod^*_{\rho\in R}T_{\alpha_\rho}$. In order to show $H\in F$, which will establish the theorem, we must show that no two $T_{\alpha_{\rho_1}}$, $T_{\alpha_{\rho_2}}$ with $\rho_1,\rho_2\in R$, $\rho_1\neq\rho_2$, are isomorphic. Suppose $T_{\alpha_{\rho_1}}\approx T_{\alpha_{\rho_2}}$. Then by Lemmas 3 and 4 and the covering property of $\{F_\gamma|\gamma\in\Gamma\}$, $\exists F_{\gamma_1}$, F_{γ_2}, F_{γ_3} satisfying:

(1) Some free factor of F_{γ_1} is conjugate to $T_{\alpha_{\rho_1}}$ in $\prod^*_{\rho\in R}T_{\alpha_\rho}$.

(2) Some free factor of F_{γ_2} is conjugate to $T_{\alpha_{\rho_2}}$ in $\prod^*_{\rho\in R}T_{\alpha_\rho}$.

(3) $F_{\gamma_1}\subset F_{\gamma_3}$ and $F_{\gamma_2}\subset F_{\gamma_3}$.

This implies that $T_{\alpha_{\rho_1}}$ and $T_{\alpha_{\rho_2}}$ are conjugate in $\prod^*_{\rho\in R}T_{\alpha_\rho}$, a contradiction since $T_{\alpha_{\rho_1}}$ and $T_{\alpha_{\rho_2}}$ are distinct free factors of $\prod^*_{\rho\in R}T_{\alpha_\rho}$.

This completes the proof.

It will be observed that the only group-theoretic property of the "incomparable" set, $\{T_\alpha | \alpha \in A\}$, of torsion abelian groups used in the proof was the each T_α was freely indecomposable and not isomorphic to any proper subgroup of itself. Since the set-theoretic lemma was proved for arbitrary ordinals, a stronger result about local sequences of groups will follow whenever a larger "incomparable" set of such freely indecomposable groups can be displayed. The author has not been able to find such a set of cardinality greater than c.

I would like to thank Dr. L. Sonneborn who offered several simplifications of my original proof and Mr. W. Stromquist whose ideas are used in several places.

Bibliography

1. A. G. Kuroš, *The theory of groups*, Vol. II, GITTL, Moscow, 1953; English transl., Chelsea, New York, 1956.

University of Kansas