A NEW PROOF OF THE PARACOMPACTNESS OF METRIC SPACES

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The purpose of this paper is to give a new proof of A. H. Stone's Theorem that every metric space is paracompact.[1] We feel that the combinatorics involved are of some interest in their own right and that our method may be useful in other contexts. The statement that a space X is paracompact means that every open cover of X has a locally finite open refinement that covers X. (An open refinement of a collection of open sets is a new collection of open sets such that each set in the new collection is contained in some set of the original collection. A cover is said to be locally finite if each point has a neighborhood intersecting only a finite number of elements in the cover.)

PROOF. (A) We shall first get a refinement such that each point is covered only a finite number of times.

To do this we will order the elements of the given cover. Call the cover $\{C_\alpha\}$. Let $R(x, m)$ be the open circle of radius $1/2^m$ about x. A *chosen* circle (with respect to C_α) is a circle $R(x, n_x+1)$ such that (i) $R(x, n_x)\subset C_\alpha$, (ii) n_x is the smallest integer for which (i) holds and (iii) $R(x, n_x)\subset C_\beta$ for some $\beta<\alpha$. Let $\tilde{C}_\alpha$ be C_α minus the closure of the union of all the chosen circles [by C minus B, I mean C intersect the complement of B]. $\{\tilde{C}_\alpha\}$ is the refinement we want for (A).

We will show that $\{\tilde{C}_\alpha\}$ is a cover. Assume we did not cover some point x. Let C_α be the first element of the original cover to contain x. $x\notin\tilde{C}_\alpha$ and, since x is not in a chosen circle, x must be a limit point of chosen circles (with respect to C_α). [Define the expansion of $R(x, m)$ to be $R(x, m-1)$.] One of the expansions of the chosen circles must contain x, for if we approach x with a sequence of chosen circles their radius can not approach 0. But if x is in the expansion of some chosen circle it must be contained in a previous C_β.

We now show that each point is contained in only a finite number of $\tilde{C}_\alpha$. If $\tilde{C}_\beta$ is to contain x then C_β must be the first element in $\{C_\alpha\}$ to contain some $R(x, m)$. If C_β is the first to contain $R(x, m)$, C_γ the first to contain $R(x, n)$ and if $n>m$ then $\gamma\leqq\beta$. Hence the $\tilde{C}_\alpha$ that contain x form a descending sequence of ordinals and therefore only a finite number are distinct.

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[1] Mary Ellen Rudin has yet another proof of this theorem which is different but just as simple as ours.

(B) We shall now get a locally finite refinement of $\{\tilde{C}_\alpha\}$. For each x let m_x be $\frac{1}{2}$ the sup of the radius of circles about x that are contained in some $\tilde{C}_\alpha$ [we can assume that each m_x is finite because otherwise our problem has a trivial solution]. Let $S(x, n)$ be the open circle of radius n about x. Let $\tilde{C}_\alpha'$ be the union of all $S(x, m_x/2)$ such that $\tilde{C}_\alpha$ is the first of the $\{\tilde{C}_\alpha\}$ to contain $S(x, m_x)$. $\{\tilde{C}_\alpha'\}$ is a locally finite cover and $\tilde{C}_\alpha' \subset C_\alpha$.

To show local finiteness, it is enough to show that if $\tilde{C}_\alpha' \cap S(x, m_x/8) \neq 0$ then $x \in \tilde{C}_\alpha$. Let us assume the contrary, i.e. for some y, $S(y, m_y/2) \cap S(x, m_x/8) \neq 0$ and that $x \notin S(y, m_y)$. Then $m_y/2 < m_x/8$ and $y \in S(x, m_x/4)$. But then $S(y, 5/2m_y) \subset S(x, m_x)$ and hence is in some $\tilde{C}_\alpha$ contradicting the maximum property of $S(y, m_y)$.

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