REGULARITY OF NET SUMMABILITY TRANSFORMS ON CERTAIN LINEAR TOPOLOGICAL SPACES

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The famous Silverman-Toeplitz theorem concerning the regularity of a matrix transform A with complex entries has undergone much study and many generalizations. Among these have been contributions by Kojima [5], Fraleigh [3], Adams [1], and Agnew [2] concerning the summability of multiple sequences. Melvin-Melvin [6] examined the case where each entry in A is a bounded linear operator on a Banach space. Recently Ramanujan [7] extended this idea to two types of linear topological spaces; namely, Fréchet spaces and locally bounded spaces. It is the purpose of this paper to generalize the results of Ramanujan by allowing the "rows" and "columns" of A to be nets of linear continuous transformations and to investigate conditions under which A will transform certain convergent nets into convergent nets.

We shall require the following notation and assumptions:

- X, Y: linear spaces over the complex numbers;
- $\mathfrak{N}$: a collection of seminorms (see [8] for definition) on X which *separates* points of X (i.e., if $x\in X-\{0\}$, then there is an $N\in\mathfrak{N}$ such that $N(x)>0$);
- $\mathfrak{M}$: a collection of seminorms on Y that separates points;
- $\mathfrak{F}$: the locally convex, Hausdorff topology on X generated by $\mathfrak{N}$;
- $\mathfrak{G}$: the locally convex, Hausdorff topology on Y generated by $\mathfrak{M}$;
- $\{D, \leqq\}$: a directed set with *finite* initial segments (i.e., if $d\in D$, then the set $\{e \mid e\in D \text{ and } e\leqq d\}$ is finite);
- C: the collection of all *bounded convergent* nets from D to X. In this context, a net will be said to be *convergent* provided it has a limit. For a general discussion of nets, see [4]. If $f\in C$ and $N\in\mathfrak{N}$, then let $N^c(f)=\sup\{N(f(d)) \mid d\in D\}$.
- $\mathfrak{N}^c$: the set of all N^c, for $N\in\mathfrak{N}$;
- $\mathfrak{F}^c$: the locally convex, Hausdorff topology on C generated by $\mathfrak{N}^c$.

We shall assume, for the remainder of the paper, that the linear topological space $\{C, \mathfrak{F}^c\}$ is *barrelled* (see [8] for definition). This

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condition is necessary and sufficient (so long as Y is nondegenerate) to insure that the following version of the Uniform Boundedness Principle holds. For a proof and related results, see Chapter IV of [8]; and, particularly, Theorem 3 on page 69.

THEOREM 1. *If $\mathfrak{I}$ is a collection of linear continuous operators from $\{C, \mathfrak{F}^c\}$ to $\{Y, \mathcal{G}\}$ with the property*

$$\text{if } f \in C \text{ and } M \in \mathfrak{M}, \text{ then } \{M(T(f)) \mid T \in \mathfrak{I}\} \text{ is bounded};$$

then, for each $M \in \mathfrak{M}$, there is a finite subset $\{N_i^c\}_{i=1}^n$ of $\mathfrak{N}^c$ and a positive number K such that

$$\text{if } f \in C \text{ and } T \in \mathfrak{I}, \text{ then } M(T(f)) \leq K \sup_{1 \leq i \leq n} N_i^c(f).$$

Observe that, if $\mathfrak{N}$ is countable and $\{X, \mathfrak{F}\}$ is complete, then $\{C, \mathfrak{F}^c\}$ is second category and thus barrelled. That is, if $\{X, \mathfrak{F}\}$ is a Fréchet space then $\{C, \mathfrak{F}^c\}$ is barrelled.

Let A be a function from $D \times D$ into the set of linear continuous functions from $\{X, \mathfrak{F}\}$ to $\{Y, \mathcal{G}\}$ (denoted hereafter by $\mathfrak{L}(X, Y)$). The $D \times D$ matrix A is said to be *convergence preserving* provided that, if $f \in C$ and $d \in D$, then the net $h(e) = \sum_{e' \leq e} A(d, e')f(e')$ is bounded and convergent in $\{Y, \mathcal{G}\}$; and the net $A(f)$ defined, for $d \in D$, by $A(f)(d) = \lim_e h(e)$ is also bounded and convergent in $\{Y, \mathcal{G}\}$. Further, if $L \in \mathfrak{L}(X, Y)$, then A is said to be *L-regular* provided A is convergence preserving and, if $f \in C$, then $A(f)$ has limit $L(\lim_d f(d))$.

The main result of this paper is the following Toeplitz theorem which generalizes Theorem 1 of [7].

THEOREM 2. *Suppose $\{Y, \mathcal{G}\}$ is complete—in the sense that every Cauchy net from D to Y is a convergent net. The function A is convergence preserving if, and only if, the following four statements are true:*

(1) *For each $M \in \mathfrak{M}$, there is a finite subset $\{N_i^c\}_{i=1}^n$ of $\mathfrak{N}^c$ and a positive number K such that, if $f \in C$ and $(d, e) \in D \times D$, then*

$$M\left(\sum_{e' \leq e} A(d, e')f(e')\right) \leq K \sup_{1 \leq i \leq n} N_i^c(f).$$

(2) *If $f \in C$ and $d \in D$, then the net $h(e) = \sum_{e' \leq e} A(d, e')f(e')$ is convergent in $\{Y, \mathcal{G}\}$.*

(3) *If $f \in C$ with limit* 0 *and $d \in D$ then the net defined, for $d_0 \in D$, by*

$$g(d_0) = \lim_e \sum_{e' \leq e; d \nleq e'} A(d_0, e')f(e'),$$

is convergent in $\{Y, \mathcal{G}\}$.

(4) *If $x \in X$, then the net defined, for $d \in D$, by*

$$k(d) = \lim_{e} \sum_{e' \leq e} A(d, e')(x),$$

is convergent in $\{Y, \mathcal{G}\}$.

PROOF. Suppose that A is convergence preserving. For each pair (d, e) in $D \times D$, let $T_{(d,e)}$ be the linear function from C to Y defined, for $f \in C$, by $T_{(d,e)}(f) = \sum_{e' \leq e} A(d, e')f(e')$. Since D has finite initial segments and A has entries in $\mathfrak{L}(X, Y)$, it is easily seen that $T_{(d,e)}$ is continuous from $\{C, \mathfrak{F}^c\}$ to $\{Y, \mathcal{G}\}$. Furthermore, if $d \in D$ and $f \in C$, then the net $h(e) = T_{(d,e)}(f)$ is bounded in $\{Y, \mathcal{G}\}$. Suppose $M \in \mathfrak{M}$ and $d \in D$. By Theorem 1, there is a finite subset $\{N_i^c(f)\}_{i=1}^n$ in $\mathfrak{N}^c$ and a positive number K such that, if $e \in D$ and $f \in C$, then

$$M(T_{(d,e)}(f)) \leq K \sup_{1 \leq i \leq n} N_i^c(f).$$

Thus, if $f \in C$,

$$M\left(\lim_{e} T_{(d,e)}(f)\right) \leq K \sup_{1 \leq i \leq n} N_i^c(f);$$

and the linear function defined, for $f \in C$, by $T_d(f) = \lim_e T_{(d,e)}(f)$ is continuous from $\{C, \mathfrak{F}^c\}$ to $\{Y, \mathcal{G}\}$. Finally, if $f \in C$, then the net $k(d) = T_d(f)$ is bounded in $\{Y, \mathcal{G}\}$ and a reapplication of Theorem 1 shows that statement (1) holds. Statement (2) is immediate from the definition. Concerning statement (3), suppose $d \in D$ and $f \in C$. If $e \in D$, let

$$\begin{aligned} f'(e) &= f(e) && \text{if } d \nleq e, \\ &= 0 && \text{if } d < e. \end{aligned}$$

Then $f' \in C$ has limit 0; and, if $(d_0, e) \in D \times D$,

$$\sum_{e' \leq e} A(d_0, e')f'(e') = \sum_{e' \leq e; d \nleq e'} A(d_0, e')f(e').$$

Statement (3) now follows. Finally, let $x \in X$ and define, for $d \in D$, $f(d) = x$. Thus $f \in C$ and statement (4) follows from statement (2) and the definition.

To prove the converse, suppose that statements (1), (2), (3) and (4) hold; $M \in \mathfrak{M}$, $\epsilon > 0$, and $f \in C$. Let $\{N_i^c\}_{i=1}^n$ be a finite subset of $\mathfrak{N}^c$ and K a positive number with the properties of statement (1). Let $d_0 \in D$ be such that, if $d \in D$ and $d_0 \leq d$, then $N_i(f(d) - \lim f) < \epsilon/4K$ for $i = 1, 2, \cdots, n$. Secondly, let $d_1 \in D$ be such that, if $d \in D$ and $d_1 \leq d$, then

(a) $$M\left(\lim_{e} \sum_{e' \leq e} A(d, e')(\lim f) - \lim_{e} \sum_{e' \leq e} A(d_1, e')(\lim f)\right) < \frac{\epsilon}{4}\cdot$$

and

(b) $$M\left(\lim_{e} \sum_{e' \leq e; d_0 \not\leq e'} A(d, e')(f(e') - \lim f) - \lim_{e} \sum_{e' \leq e; d_0 \not\leq e'} A(d_1, e')(f(e') - \lim f)\right) < \frac{\epsilon}{4}\cdot$$

If $d \in D$ and $d_1 \leq d$, then

$$M\left(\lim_{e} \sum_{e' \leq e} A(d, e')f(e') - \lim_{e} \sum_{e' \leq e} A(d_1, e')f(e')\right) < \epsilon.$$

Thus, since $\{Y, \mathcal{G}\}$ is net complete, A is convergence preserving and Theorem 2 is proved.

An analogous argument will yield a proof of the following theorem where $L \in \mathcal{L}(X, Y)$.

THEOREM 3. *The function A is L-regular if, and only if, statements* (1) *and* (2) *of Theorem 2 hold and the following two statements also hold:*

(3′) *If $f \in C$ with limit* 0 *and $d \in D$ then the net defined, for $d_0 \in D$, by*

$$g(d_0) = \lim_{e} \sum_{e' \leq e; d \not\leq e'} A(d_0, e')f(e'),$$

is convergent to 0 *in* $\{Y, \mathcal{G}\}$.

(4′) *If $x \in X$, then the net defined, for $d \in D$, by*

$$k(d) = \lim_{e} \sum_{e' \leq e} A(d, e')(x),$$

is convergent to $L(x)$ in $\{Y, \mathcal{G}\}$.

It should be noted that in Theorem 3 we need not require that $\{Y, \mathcal{G}\}$ be net complete.

We see that Theorems 2 and 3 are extensions of Theorems 1 and 2 of [7], as well as the usual Silverman-Toeplitz theorems. However, there is one application which is important in the study of functions of several complex variables that is included in the present theory but not included in [7]. In particular, suppose that each of $\{X, \mathcal{F}\}$ and $\{Y, \mathcal{G}\}$ is the complex plane with the usual topology, k is a positive integer, and D is the set of k-tuples of positive integers where, if each of n and m is in D, then $n < m$ provided that, for $i = 1, 2, \cdots, k$, $n(i) < m(i)$. In this case, letting I denote the identity function on X, we have the following theorem (of course, in this setting, A is simply a $D\times D$ matrix of complex numbers).

THEOREM 4. *The function A is I-regular if, and only if, the following three statements are true:*

(1) *There is a positive number K such that, if $(d, e)\in D\times D$, then $\sum_{e'\leq e}|A(d, e')|\leq K$.*

(2) *If $d\in D$, then the complex number net defined, for $d_0\in D$, by*

$$g(d_0) = \lim_e \sum_{e'\leq e;\, d\not\leq e'} |A(d_0, e')|$$

is convergent to 0.

(3) *The net defined, for $d\in D$, by*

$$k(d) = \lim_e \sum_{e'\leq e} A(d, e')$$

has limit 1.

PROOF. In this context, statements (1) and (2) of Theorem 2 are equivalent to statement (1). Also statement (4′) of Theorem 3 is equivalent to statement (3). As statement (2) implies statement (3′) of Theorem 3, all that remains is to show that if A is I-regular, then statement (2) holds. Hence, suppose that A is I-regular, $d\in D$; and for each $d_0\in D$, let

$$g(d_0) = \lim_e \sum_{e'\leq e;\, d\not\leq e'} |A(d_0, e')|.$$

Suppose further that g does not have limit 0, and let $\epsilon>0$ be such that, if $e\in D$, then there is an $>e'$ with the property that $g(e')>\epsilon$. Let $S=\{d' \mid d'\in D, d\not\leq d'\}$, $e_1\in D$ such that $e_1>1$ (the constant 1 member of D) and K_1 a finite subset of S such that

$$\sum_{e\in S} |A(e_1, e)| - \sum_{e\in K_1} |A(e_1, e)| < \frac{\epsilon}{4}\cdot$$

Suppose p is a positive integer and disjoint finite subsets K_1, K_2, $\cdots K_p$ of S have been chosen, along with elements $e_1, e_2, \cdots, e_p$ of D. Using the fact that if $e\in D$, then the net $A(\cdot, e)$ has limit 0; let $e_{p+1}\in D$ be such that

$$e_p < e_{p+1}, \qquad \sum_{e\in \cup_{i=1}^p K_i} |A(e_{p+1}, e)| < \frac{\epsilon}{4}, \qquad \text{and} \quad g(e_{p+1}) > \epsilon.$$

Let K_{p+1} be a finite subset of S such that $K_{p+1}\cap \bigcup_{i=1}^p K_i=\emptyset$ and

$$\sum_{e\in S} |A(e_{p+1}, e)| - \sum_{e\in K_{p+1}} |A(e_{p+1}, e)| < \frac{\epsilon}{4}\cdot$$

Define the member f of C as follows: if $e\in D$, then

$$\begin{aligned} f(e) &= |A(e_n, e)|/A(e_n, e) \qquad \text{if } e \in K_n \text{ and } A(e_n, e) \neq 0, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

If p is a positive integer, then

$$\begin{aligned} \left|\lim_e \sum_{e'\leqq e} A(e_p, e')f(e')\right| &\geqq \sum_{e'\in K_p} |A(e_p, e')| - \sum_{e'\in S-K_p} |A(e_p, e')| \\ &= \sum_{e'\in S} |A(e_p, e')| - 2\sum_{e'\in S-K_p} |A(e_p, e')| \\ &> \epsilon - \epsilon/2 = \epsilon/2. \end{aligned}$$

Since the sequence e is cofinal in D, we see that $A(f)$ does not have limit 0. This contradiction establishes Theorem 4.

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