POSITIVE HARMONIC FUNCTIONS OF A BRANCHING PROCESS

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1. **Introduction.** Let $\{p(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers, one defines an infinite matrix $p(x, y)$ $(x, y) \in N \times N$ as follows:

$$p(0, y) = \delta(0, y), \qquad p(1, y) = p(y);$$

the other lines of the matrix are such that

$$p(x + 1, y) = \sum_{y_1+y_2=y} p(1, y_1)p(x, y_2).$$

We will say that the matrix $p(x, y)$ is the branching matrix corresponding to the sequence $\{p(n)\}_{n=0}^{\infty}$. When $\sum_{n=0} p(n) = 1$, the matrix is stochastic and describes a Markov process which is called a branching process.

We would like to get information about the positive harmonic functions $h(x)$:

$$\sum_y p(x, y)h(y) = h(x), \qquad h(x) \geqq 0.$$

We will see that there is a one-to-one linear correspondence between these harmonic functions and those of a second triangular branching matrix. From that, it follows that there can be only one extreme harmonic function equal to 1 at $x = 0$. This function whenever it exists is an exponential function. We will be able to describe all the harmonic functions only when $\sum_{n=0}^{\infty} p(n) = \sum_{n=0}^{\infty} np(n) = 1$. This case has been already solved by Kesten, Ney and Spitzer [1].

2. **Functional meaning of harmonic functions.** Let $h(x)$ be a positive function defined on N and let $p(x, y)$ be the branching matrix corresponding to a sequence $\{p(n)\}_{n=0}^{\infty}$ of positive numbers, let us consider the cone of formal power series in a variable z with positive coefficients. On this cone, we define the following functional:

$$H(g(z)) = \sum_{x=0}^{\infty} b(x)h(x) \qquad \text{if } g(z) = \sum_{x=0}^{\infty} b(x)z^x.$$

If $f(z) = \sum_{n=0}^{\infty} p(n)z^n$, we have that $(f(z))^x = \sum_{y=0}^{\infty} p(x, y)z^y$. Since $h(x) = H(z^x)$, $h(x)$ is harmonic precisely when $H(z^x) = H((f(z))^x)$, that is to say when $H(g(z)) = H(g(f(z)))$ for every power series $g(z)$.

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3. Transformation of harmonic functions.

THEOREM. (a) *If there is no number* $a \geqq 0$ *such that* $\sum_{n=0}^{\infty} p(n)a^n$ *converges to* a, *there is no harmonic function different from* 0.

(b) *If* a *is the smallest positive solution of* $\sum_{n=0}^{\infty} p(n)t^n = t$, *if* $p'(0) = 0$ *and*

$$p'(n) = \sum_{m=n}^{\infty} \binom{m}{n} p(m) a^{m-n} \qquad (n \geqq 1),$$

and if $p'(x, y)$ *is the branching matrix corresponding to the sequence* $\{p'(n)\}$, *there is a linear one-to-one correspondence between positive harmonic functions of the matrix* $p(x, y)$ *and those of the matrix* $p'(x, y)$. *Such a mapping is*

$$h'(x) = \sum_{v=0}^{x} \binom{x}{v} (-a)^{x-v} h(v), \qquad h(x) = \sum_{v=0}^{x} \binom{x}{v} a^{x-v} h'(v).$$

PROOF. There is something to be proved only when $p(0) \neq 0$. Let $f_n(z)$ be the functional product of the power series $f(z)$ n times with itself. Let us start with a harmonic function $h(x)$ for the matrix $p(x, y)$. If $g(z) = \sum_{x=0}^{\infty} b(x)z^x$ is a power series such that $\sum_{x=0}^{\infty} |b(x)| h(x) < \infty$, we will set $H(g(z)) = \sum_{x=0}^{\infty} b(x)h(x)$. Let n be an integer and t be a number between 0 and $f_n(0)$, we set $h_t(x) = H((z-t)^x)$. Since $f_n(z) - t$ is a power series with positive coefficients, we get that $h_t(x) = H((f_n(z)-t)^x) \geqq 0$.

(a) If $\lim_{n\to\infty} f_n(0) = \infty$, then for every t,

$$h_t(x) = \sum_{v=0}^{x} \binom{x}{v} (-t)^{x-v} h(v) \geqq 0.$$

So for every x, $h(x) = 0$.

(b) If $\lim_{n\to\infty} f_n(0) < \infty$, $a = \lim_{n\to\infty} f_n(0)$. We will set $h'(x) = H((z-a)^x)$.

Since $h'(x) = \lim_{t\uparrow a} h_t(x)$, $h'(x) \geqq 0$. On the cone of power series in a variable w $(w = z - a)$ with positive coefficients, $h'(x)$ defines a positive functional H', where $H'(w^x) = h'(x)$.

$$H'((f(a + w) - a)^x) = H((f(z) - a)^x) = H((z - a)^x) = H'(w^x).$$

Since $f(a+w) - f(a) = \sum_{n=0}^{\infty} p'(n)w^n$, $h'(x)$ is a harmonic function for the matrix $p'(x, y)$.

Conversely, if $h'(x)$ is a positive harmonic function for the matrix $p'(x, y)$ and if H' is the corresponding functional on the power series in w, we set $h(x) = H'((a + w)^x)$. From the change of variable $w \to f(a + w) - a$, which leaves H' unchanged, we get that $h(x) = H'((f(a+w))^x)$. So $h(x)$ is also a harmonic function and

$$h(x) = \sum_{v=0}^{x} \binom{x}{v} a^{x-v} h'(v).$$

4. **Applications.** Let us remark that the matrix $p'(x, y)$ is triangular: if $y < x$, $p'(x, y) = 0$. On the diagonal $p'(x, x) = \mu^x$ where $\mu = \sum_1^\infty mp(m)a^{m-1}$. μ is the derivative of $f(z)$ at $z = a$, so $\mu \leqq 1$. If $h'(x)$ is an extreme function such that $h'(0) = 1$, since $\delta(0, x)$ is harmonic and from the fact that $h'(x) = [h'(x) - \delta(0, x)] + \delta(0, x)$ is the sum of two positive harmonic functions, then $h'(x) = \delta(0, x)$. The corresponding harmonic function $h(x)$ is a^x. This exponential is the only extreme harmonic function taking the value 1 at $x = 0$ for the matrix $p(x, y)$. This last fact could have been proven directly. Let us sketch two other proofs for that.

Let $h(x)$ be an extreme harmonic function for $p(x, y)$ such that $h(0) = 1$.

(a) Let $\tilde{h}(x) = \min \{h(y_1)h(y_2) \cdots h(y_x) \mid y_1 + y_2 + \cdots + y_x = x\}$; we will check that $\tilde{h}(x)$ is a superharmonic function. Let $w_1, w_2, \cdots, w_x$ be x nonnegative integers whose sum is x, then

$$\begin{aligned}
&\sum_y p(x, y)\tilde{h}(y) \\
&\qquad = \sum_{z_1, z_2, \cdots, z_x} p(w_1, z_1)p(w_2, z_2) \cdots p(w_x, z_x)\tilde{h}(z_1 + z_2 + \cdots + z_x) \\
&\qquad \leqq \sum_{z_1, z_2, \cdots, z_x} p(w_1, z_1) \cdots p(w_x, z_x)h(z_1) \cdots h(z_x) \\
&\qquad \leqq h(w_1)h(w_2) \cdots h(w_x).
\end{aligned}$$

Hence $\sum_y p(x, y)\tilde{h}(y) \leqq \tilde{h}(x)$.

Using the Riesz decomposition theorem for superharmonic function and noticing that in our case there is no potential $U(x)$ such that $U(0) = 1$, we get that $\tilde{h}(x)$ is harmonic and is equal to $h(x)$.

(b) $\delta(0, x)$ is a subharmonic function which is smaller than $h(x)$, so $h(x) = \lim_{n\to\infty} \sum_{v=0}^\infty p_n(x, y)\delta(0, y) = \lim_{n\to\infty} (f_n(0))^x$ where $p_n(x, y)$ is the branching process corresponding to the power series $f_n(z)$.

If $\mu = 1$ and if $h'(x)$ is a harmonic function such that $h'(1) = 0$, one can check easily that $h'(x) = 0$ if $x > 0$. So the only positive harmonic functions are in that case $A\delta(0, x) + B\delta(1, x)$ where $A \geqq 0$, $B \geqq 0$.

Reference

1. H. Kesten, P. Ney and F. Spitzer, *The Galton-Watson process with mean one and finite variance*, Teor. Verojatnost. i Primenen. **11** (1966), 579–611.

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