A NOTE ON A THEOREM OF JACOBSON

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The question as to whether every derivation of a simple algebra is inner, is still unsettled. The simple proofs given below of Theorem A, which is a special case of a well-known theorem (see [**4**, pp. 22–23]), and that of Theorem B would possibly be a new approach to this question.

THEOREM A. *If A is a simple algebra with identity over an algebraically closed field F of characteristic zero, then every derivation of A is inner.*

PROOF. In the first instance, let A have an identity or not. Let R_x (L_x) denote the right (left) multiplication in A, and D be a derivation of A. Then A has neither proper ideals nor proper D-ideals. In other words, the Lie algebra L (L') generated by R_x, L_x (R_x, L_x and the derivation D) is irreducible. Moreover, $L'=L+\{\alpha D\}_{\alpha\in F}$ (a vector space sum and not necessarily a direct sum), and L is an ideal of L'. Further, by a theorem of Jacobson [**1**, p. 47], we have $L=C\oplus[LL]$; $L'=C'\oplus[L'L']$ for centers C, C' of L, L'; $[LL]$, $[L'L']$ are semisimple; $[LL]$ is an ideal of $[L'L']$. Any transformation T in C commutes with the irreducible associative algebra generated by R_x, L_x and hence should be a multiple of the identity transformation I, by Schur's lemma. Now, if A contains an identity, $C=FI$, $C'=FI$; since the dimension of L' is at the most greater by unity than that of L, and since $[L'L']$ cannot have a one dimensional (abelian) ideal complementary to $[LL]$, $[LL]=[L'L']$; i.e., $L=L'$, or, $D\in L$. Thus, every derivation of A is inner.

REMARK 1. In case A is any simple nonassociative algebra, then $C\subseteq C'$ and therefore $[LL]=[L'L']$ in this case also. If, in addition $C=C'$ for every derivation D of A, then every derivation of A will be inner. Because F is algebraically closed, $C=0$ or $C=FI$, $C'=0$ or $C'=FI$. Since $C=FI$ implies $C'=FI=C$ and since every derivation is inner in this case as well as when $C'=0$, the question raised at the outset boils down to the consideration of the only case $C=0$, $C'=FI$. The plausibility of this case remains to be seen.

Now, in the case of simple Lie algebra A over a field F of characteristic zero, $L=\{\text{ad } x\}_{x\in A}$; $L'=L+\{\alpha D\}_{\alpha\in F}$. Since A is simple, the center of $A=\{x\in A \mid \text{ad } x=0\}=\{0\}$. If ad $y\in$center C of L, then

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$[\mathrm{ad}\ x, \mathrm{ad}\ y] = \mathrm{ad}[x, y] = 0$ for every x in A; hence $[x, y] = 0$, i.e., $\mathrm{ad}\ y = 0$ and $C = 0$. If $\mathrm{ad}\ z + \alpha D$ belongs to the center C' of L', then $[\mathrm{ad}\ z + \alpha D, \mathrm{ad}\ x] = 0$ for all x, i.e., $\mathrm{ad}[z, x] - \alpha\ \mathrm{ad}\ xD = 0$ for all x in A, or, $[z, x] - \alpha xD = 0$, i.e., $-(\mathrm{ad}\ z + \alpha D) = 0$. Hence $C' = 0$. Thus L, L' are semisimple [**1**, p. 47]. Further arguments as in the proof of Theorem A yield

THEOREM B. *Every derivation of a simple Lie algebra A over a field of characteristic zero is inner.*

REMARK 2. Theorem B is, more generally, true for a simple Malcev algebra A [**3**]. For, the center C of L is then known to be the zero ideal [**2**, Corollary 5.32]. A similar argument shows that the center C' of L' is also the zero ideal.

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