FINITELY GENERATED COHERENT ANALYTIC SHEAVES

BERNARD KRIPKE

If $\mathcal{S}$ is a coherent analytic sheaf on the complex analytic space X, then for each $x \in X$, the stalk $\mathcal{S}(x)$ is a finitely generated $\mathcal{O}(x)$-module, where $\mathcal{O}$ is the structure sheaf of X [1]. Since $\mathcal{O}(x)$ is a local ring, there is a minimum number, $\#(\mathcal{S}, x)$, of germs that generate $\mathcal{S}(x)$ as an $\mathcal{O}(x)$-module, and every set of generators for $\mathcal{S}(x)$ contains a subset of $\#(\mathcal{S}, x)$ generators [2, p. 14].

If there are n global sections $s_1, \cdots, s_n \in \mathcal{S}(X)$ whose germs generate the stalk of $\mathcal{S}$ at every point, then evidently:

(A) for every $x \in X$, $\mathcal{S}(x)$ is generated by global sections of $\mathcal{S}$ and,

(B) $\{\#(\mathcal{S}, x) : x \in X\}$ is a bounded set of integers. In fact, $\{\#(\mathcal{S}, x) : x \in X\}$ is bounded by n. The principal result of this note is that the converse also is true in case X has finite global dimension. If the global sections of $\mathcal{S}$ generate its stalk at each point and if $\{\#(\mathcal{S}, x) : x \in X\}$ is bounded, then *finitely many* of the global sections of $\mathcal{S}$ generate its stalk at each point.

Let us say that a subset G of $\mathcal{S}(X)$ *generates* $\mathcal{S}|K$ if for each $x \in K$, $\{s(x) : s \in G\}$ generates the stalk $\mathcal{S}(x)$. If $K = X$, say that G *generates* $\mathcal{S}$. An ordered n-tuple $(s_1, \cdots, s_n) \in \mathcal{S}(X^n)$ generates $\mathcal{S}|K$ if $\{s_1, \cdots, s_n\}$ generates $\mathcal{S}|K$. Let $G(\mathcal{S}, n, K)$ be the set of all n-tuples in $\mathcal{S}(X)^n$ which generate $\mathcal{S}|K$.

If U is an open subset of X, then $\mathcal{S}(U)$ has a natural metrizable topology, which makes $\mathcal{S}(U)$ into a Fréchet space. If V is open and contains U, the restriction map $r_{VU} : \mathcal{S}(V) \to \mathcal{S}(U)$ is continuous [1, Chapter VIII]. A *residual set* in $\mathcal{S}(X)^n$ is the complement of a set of the first category.

1. THEOREM. *Let X be a d-dimensional analytic space and let $\mathcal{S}$ be a coherent analytic sheaf on X that is generated by $\mathcal{S}(X)$. If $\#(\mathcal{S}, x) \leq n$ for every $x \in X$, then $G(\mathcal{S}, n(d+1), X)$ is a dense residual set in $\mathcal{S}(X)^{n(d+1)}$; in particular, it is not empty.*

The theorem follows from a series of lemmas.

2. LEMMA. *Let X be a complex analytic space and let $\mathcal{S}$ be a coherent analytic sheaf on X. If U is an Oka-Weil domain in X [1, p. 211] and K is a compact $\mathcal{O}(U)$-convex subset of U, then $G(\mathcal{S}, n, K)$ is open in $\mathcal{S}(X)^n$, for each positive integer n.*

Received by the editors February 26, 1968.

PROOF. Suppose $t_1, \cdots, t_n \in \mathcal{S}(X)$ generate $\mathcal{S}|K$. Then K has an open neighborhood $V \subseteq U$ that is also an Oka-Weil domain such that $t_1, \cdots, t_n$ generate $\mathcal{S}|V$ [1, pp. 211 and 244]. Identify $\mathcal{C}^{n \times n}$ with the space of $n \times n$ complex matrices, and let $E: \mathcal{O}^{n \times n}|V \to \mathcal{S}^n|V$ be the map defined by $E(f) = (\sum_j f_{1j}t_j, \cdots, \sum_j f_{nj}t_j)$. Then E is a homomorphism of coherent sheaves. Ker E is a coherent sheaf, and by Cartan's Theorem B, $H^1(V, \text{Ker } E) = 0$. Since $t_1, \cdots, t_n$ generate $\mathcal{S}|V$, the sequence $0 \to H^0(V, \text{Ker } E) \to H^0(V, \mathcal{O}^{n \times n}) \to H^0(V, \mathcal{S}) \to 0$ is exact. That is, $E: \mathcal{O}(V)^{n \times n} \to \mathcal{S}(V)$ is a surjection. $R = \{f \in \mathcal{O}(V)^{n \times n} : f(x)$ is an invertible matrix for each $x \in K\}$ is open in $\mathcal{O}(V)^{n \times n}$, since the topology of $\mathcal{O}(V)^{n \times n}$ is that of uniform convergence on compacta [1, p. 237], and the set of invertible matrices is open in $\mathcal{C}^{n \times n}$. It follows from the Open Mapping Theorem for Fréchet spaces that $E(R)$ is open in $\mathcal{S}(V)^n$. Since r_{XV} is continuous, $r_{XV}^{-1}(E(R))$ is open in $\mathcal{S}(X)^n$. But if $s \in \mathcal{S}(X)^n$ and $r_{XV}(s) \in E(R)$, then s generates $\mathcal{S}|V$. Thus $G(\mathcal{S}, n, K)$ contains a neighborhood $r_{XV}^{-1}(E(R))$ of t.

3. LEMMA. *Let X be a complex analytic space, $\mathcal{S}$ a coherent analytic sheaf on X, and $s_1, \cdots, s_n \in \mathcal{S}(X)$. Then $Y = \{y \in X : s_1(y), \cdots, s_n(y)$ do not generate $\mathcal{S}(y)\}$ is an analytic subvariety of X.*

PROOF. Let $\mathfrak{J}$ be the subsheaf of $\mathcal{S}$ generated by $s_1, \cdots, s_n$. Then Y is the support of the coherent analytic sheaf $\mathcal{S}/\mathfrak{J}$ [3, p. 87].

4. LEMMA. *Let X be a complex analytic space, let $x \in X$, and let $\mathcal{S}$ be a coherent analytic sheaf on X such that $\mathcal{S}(X)$ generates $\mathcal{S}(x)$. Let $n \geqq \#(\mathcal{S}, x)$. Then $G(\mathcal{S}, n, \{x\})$ is dense in $\mathcal{S}(X)^n$.*

PROOF. Since $\mathcal{S}(X)$ generates $\mathcal{S}(x)$ and $n \geqq \#(\mathcal{S}, x)$, we can choose a $t \in \mathcal{S}(X)^n$ that generates $\mathcal{S}(x)$. Let s be any element of $\mathcal{S}(X)^n$. Say $s_i(x) = \sum_j c_{ij}t_j(x)$, where the matrix-valued function c is analytic in a neighborhood of x. Then $s(x) - \lambda t(x) = (c - \lambda I)t(x)$, so that $s - \lambda t$ will generate $\mathcal{S}(x)$ provided that the matrix $c - \lambda I$ is nonsingular in a neighborhood of x. This will be true if λ is distinct from each of the n eigenvalues of the matrix $c(x)$. There are arbitrarily small numbers λ with this property. Hence there are sections $s - \lambda t$ of $\mathcal{S}(X)^n$ arbitrarily close to s that generate $\mathcal{S}(x)$.

5. LEMMA. *Let X be a d-dimensional complex analytic space, let $\mathcal{S}$ be a coherent analytic sheaf on X, and let K be a compact subset of X. Suppose that $\mathcal{S}(X)$ generates $\mathcal{S}|K$ and that $n \geqq \#(\mathcal{S}, x)$ for each $x \in K$. Then $G(\mathcal{S}, n(d+1), K)$ is dense in $\mathcal{S}(X)^{n(d+1)}$.*

PROOF. Let A be a nonempty open set in $\mathcal{S}(X)^{n(d+1)}$. Then A contains a nonempty open set of the form $A_1 \times \cdots \times A_{d+1}$, where A_i

is open in $\mathcal{S}(X)^n$ for $i=1, \cdots, d+1$. Suppose that $0 \leqq k \leqq d+1$. Let us show that for each i such that $1 \leqq i \leqq k$, we can choose a section $s^i \in A_i$ with the following property. Let $Y_k = \{x \in X: (s^1(x), \cdots, s^k(x))$ does not generate $\mathcal{S}(x)\}$. (Y_k is a variety by Lemma 3.) Then no irreducible branch of Y_k of dimension greater than $d-k$ intersects K.

The proof is by induction on k. For $k=0$, $Y_0 = X$ and there is nothing to prove. Suppose we have chosen $s^1, \cdots, s^k$ so that no irreducible branch of Y_k of dimension greater than $d-k$ intersects K. Let $B_1, \cdots, B_p$ be the irreducible branches of Y_k which do intersect K. Since $\mathcal{S}$ is coherent, there is actually a neighborhood U of K such that $\#(\mathcal{S}, x) \leqq n$ for each $x \in U$. Therefore, for each $j=1, \cdots, p$, we can choose a regular point $x_j \in B_j$ such that $\#(\mathcal{S}, x_j) \leqq n$. Then $D_j = \{s \in \mathcal{S}(X)^n: s$ generates $\mathcal{S}(x_j)\}$ is open (Lemma 2) and dense (Lemma 4) in $\mathcal{S}(X)^n$. Choose s^{k+1} in $A_{k+1} \cap D_1 \cap \cdots \cap D_p$. Then no $(n-k)$-dimensional branch of Y_{k+1} can intersect K.

In particular, $(s^1, \cdots, s^{d+1})$ is an element of A such that $Y_{d+1} \cap K = \emptyset$, or in other words, $(s^1, \cdots, s^{d+1})$ is an element of A that generates $\mathcal{S} | K$.

Proof of Theorem 1. Express X as the union of countably many compact subsets $K_1, K_2, K_3, \cdots$, each of which is contained in an Oka-Weil domain in which it is holomorphically convex. According to Lemmas 2 and 5, $G(\mathcal{S}, n(d+1), K_j)$ is open and dense in $\mathcal{S}(X)^{n(d+1)}$ for each $j=1, 2, 3, \cdots$. Therefore,

$$G(\mathcal{S}, n(d+1), X) = \bigcap_{j=1}^{\infty} G(\mathcal{S}, n(d+1), K_j)$$

is a residual set in $\mathcal{S}(X)^{n(d+1)}$. Since $\mathcal{S}(X)^{n(d+1)}$ is a Fréchet space, the Baire Category Theorem shows that $G(\mathcal{S}, n(d+1), X)$ is dense in $\mathcal{S}(X)^{n(d+1)}$.

6. Corollary. *If X is a d-dimensional analytic space and B is a n-dimensional vector bundle over X which is generated by its global sections, then B is generated by $n(d+1)$ of its global sections.*

If X is a Stein space, every coherent sheaf on X satisfies condition (A), according to Cartan's Theorem A [1]. At least when X is an open subset of a Stein manifold, the converse is also true. Indeed, in this case X satisfies the hypotheses of the following proposition, according to Rossi [4].

7. Proposition. *Let X be an analytic space with the following properties.*

(a) *X can be embedded as an open subset of a Stein space Y in such a way that the restriction map r: $\mathcal{O}(Y)\to\mathcal{O}(X)$ is bijective.*

(b) *Whenever $\mathcal{S}$ is the sheaf of ideals of a 0-dimensional variety in X, $\mathcal{S}(X)$ generates $\mathcal{S}$.*

Then X is a Stein space.

PROOF. It will be enough to show that $X=Y$. If $X\neq Y$, there must be a component C of Y which is not contained in X. However, $C\cap X$ cannot be empty since the restriction $\mathcal{O}(Y)\to\mathcal{O}(X)$ is injective. Therefore, there must be a y on the boundary of $C\cap X$ with respect to C.

Let $\{x_n\}$ be a sequence in $C\cap X$ converging to y, and let $\mathcal{S}$ be the sheaf of ideals of $\{x_n\}$ over X. Let $Z=\{x\in X: f(x)=0$ for each $f\in\mathcal{S}(X)\}$. It will be enough to show that Z has dimension greater than 0, for then we will have contradicted the hypothesis that $\mathcal{S}$ is generated by its global sections. Since r is a bijection, we can form $W=\{w\in Y: (r^{-1}f)(w)=0$ for each $f\in\mathcal{S}(X)\}$. Clearly, W is a subvariety of Y and $W\cap X=Z$. Since y is an accumulation point of Z, $\dim_y(W)\neq 0$. If the irreducible branches of W passing through y are $B_1, \cdots, B_p$, at least one of them must contain infinitely many points of Z, B_1 let us say. But then $\dim B_1>0$, and $\dim Z\geqq\dim B_1$.

8. PROPOSITION. *Let M be a 2-dimensional complex manifold and let f: $\mathcal{O}^n\to\mathcal{O}^m$ be an $\mathcal{O}$-homomorphism. Then for every $x\in M$, $\#(\operatorname{Ker} f, x)\leqq n$.*

PROOF. Consider the exact sequence $(\operatorname{Ker} f)(x)\to\mathcal{O}(x)^n\xrightarrow{f}\mathcal{O}(x)^m\to\mathcal{O}(x)^m/(\operatorname{Im} f)(x)\to 0$. According to the Hilbert Syzygy Theorem [1, p. 74], $(\operatorname{Ker} f)(x)$ is a free $\mathcal{O}(x)$-module. Since $(\operatorname{Ker} f)(x)$ is a free submodule of $\mathcal{O}(x)^n$, $\#(\operatorname{Ker} f, x)\leqq n$. (The field of quotients Q of $\mathcal{O}(x)$, being the direct limit of copies of $\mathcal{O}(x)$, is a flat $\mathcal{O}(x)$-module. Thus the sequence $0\to Q\otimes_{\mathcal{O}(x)}(\operatorname{Ker} f)(x)\to Q\otimes_{\mathcal{O}(x)}\mathcal{O}(x)^n$ is exact. But $Q\otimes\mathcal{O}(x)^n$ is an n-dimensional vector space over Q, and $Q\otimes(\operatorname{Ker} f)(x)$ is a vector space over Q of dimension $\#(\operatorname{Ker} f, x)$.)

9. COROLLARY. *Let M be a 2-dimensional Stein manifold. If I and J are finitely generated ideals in the ring of holomorphic complex-valued functions on M, then $I\cap J$ is also finitely generated.*

PROOF. The following proof was suggested to me by Lance Small. Let $a_1, \cdots, a_n$ be generators for I, and let $b_1, \cdots, b_m$ be generators for J. Let $\mathcal{I}$ and $\mathcal{J}$ be respectively the subsheaves of $\mathcal{O}$ generated by I and J. According to Cartan's Theorem B, $I=\mathcal{I}(M)$, $J=\mathcal{J}(M)$, and $I\cap J=(\mathcal{I}\cap\mathcal{J})(M)$. Thus it will suffice to prove that $\mathcal{I}\cap\mathcal{J}$ is generated by finitely many of its global sections. In fact, it will be shown that

$\#(\mathcal{I}\cap\mathcal{J}, x)\leqq n+m$ for every $x\in M$, and hence that $I\cap J$ is generated by $3(n+m)$ of its elements according to Theorem 1.

Let $f\colon \mathcal{O}^{n+m}\to\mathcal{O}$ be the map defined by $f(c_1, \cdots, c_{n+m})=c_1a_1+\cdots+c_na_n-c_{n+1}b_1-\cdots-c_{n+m}b_m$. Then $\#(\operatorname{Ker} f, x)\leqq n+m$ for every $x\in M$ by Proposition 8. But the formula $\pi(c_1, \cdots, c_{n+m})=c_1a_1+\cdots+c_na_n$ evidently defines a surjection $\pi\colon \operatorname{Ker} f\to\mathcal{I}\cap\mathcal{J}$. It follows that $\#(\mathcal{I}\cap\mathcal{J}, x)\leqq\#(\operatorname{Ker} f, x)\leqq n+m$ for every $x\in M$.

I do not know whether Corollary 9 would remain true if the condition of 2-dimensionality were dropped. I conjecture that it would not. In one dimension, the corollary is trivial since every finitely generated ideal is then principal.

10. Example. The bound $n(d+1)$ is the best possible if the density of $G(\mathcal{S}, n(d+1), X)$ in $\mathcal{O}(X)^{n(d+1)}$ is to be preserved.

Consider, in the complex plane $\mathcal{C}$, the subsheaf $\mathcal{S}$ of $\mathcal{O}$ generated by the coordinate function z. Although one global section suffices to generate $\mathcal{S}$, every section in a neighborhood of the section z^2 has two zeros in $\mathcal{C}$, and therefore fails to generate $\mathcal{S}$.

References

1. R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

2. M. Nagata, *Local rings*, Interscience, New York, 1962.

3. R. Narasimhan, *Introduction to the theory of analytic spaces*, Lecture notes in mathematics No. 25, Springer-Verlag, New York, 1966.

4. H. Rossi, *On envelopes of holomorphy*, Comm. Pure Appl. Math. **16** (1963), 9–19.

University of California, Berkeley