ON AN INEQUALITY OF NEHARI

A. M. FINK AND D. F. ST. MARY

Nehari [1, Theorem I] claims that if $[a, b]$ contains n zeros of a nontrivial solution of $y^{(n)}+p_n y^{(n-1)}+\cdots+p_1 y=0$, then

$$\sum_{k=1}^{n} 2^k (b-a)^{n-k} \int_a^b |p_k| > 2^{n+1}. \tag{1}$$

In a private communication with one of the authors, Professor Nehari has indicated that the inequality is undecided since the argument given in [1] that Theorem II implies Theorem I is incorrect. It is the purpose of this note to show that (1) is correct for $n=2$. In fact, we prove a stronger result for

$$y'' + gy' + fy = 0. \tag{2}$$

THEOREM. *Let a and b be successive zeros of a nontrivial solution to (2) where f and g are integrable. Then*

$$(b-a)\int_a^b f^+(x)dx - 4\exp\left(-\frac{1}{2}\int_a^b |g|(x)dx\right) > 0 \tag{3}$$

and a fortiori

$$(b-a)\int_a^b f^+(x)dx + 2\int_a^b |g|(x)dx > 4. \tag{4}$$

If a and b are successive zeros, then there is a $c\in(a, b)$ such that $y'(c)=0$. Nehari shows that

$$1 < (c-a)\int_a^c |f| + \int_a^c |g|, \tag{5}$$

and a similar inequality for the interval (c, b). The trick is to combine the two to get (4).

We start with an inequality which is stronger than (5). Consider the equation $(ry')'+py=0$ for $r>0$, with r and p integrable.

LEMMA (SEE [2]). *If $y(a)=0$ and $y'(c)=0$, $a<c$, then*

$$1 < \int_a^c r^{-1} \int_a^c p^+$$

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where $p^+(x) = \max\{p(x), 0\}$.

PROOF. Let $|y(x)| = \max|y(t)|$. Then

$$(y(x))^2 \leqq \left(\int_a^c |y'|\right)^2 = \left(\int_a^c r^{-1/2}|\sqrt{r}\,y'|\right)^2$$
$$< \left(\int_a^c r^{-1}\right)\int_a^c r(y')^2 = \left(\int_a^c r^{-1}\right)\int_a^c py^2$$
$$\leqq \left(\int_a^c r^{-1}\right)\int_a^c p^+y^2 \leqq y^2(x)\int_a^c r^{-1}\int_a^c p^+$$

from which the result follows. We have used the Schwarz inequality and the identity $\int_a^c r(y')^2 = \int_a^c py^2$ which may be verified by an integration by parts.

A similar result holds on (c, b) where $y(b) = 0$. Now we want to apply the lemma to the equation (2). Taking $r = \exp \int_a^c g$ and $p = rf$ we get

$$(6) \quad 1 < \int_a^c \exp\left(-\int_a^t g(s)ds\right)dt \int_a^c f^+(x)\exp\left(\int_a^x g(s)ds\right)dx.$$

Writing this as a double integral we see that

$$1 < \int_a^c\int_a^c \exp\left(\int_t^x g(s)ds\right)f^+(x)dxdt$$
$$< \int_a^c\int_a^c \exp\left(\int_a^c |g|(s)ds\right)f^+(x)dxdt$$

or

$$(7) \qquad 1 < (c-a)\int_a^c f^+(x)dx \exp\int_a^c |g|(s)ds.$$

A similar result holds for the interval (c, b) where $y(b) = 0$.

In order to motivate a later inequality we observe that (7) implies (5) with $|f|$ replaced by f^+. Indeed, letting $A_0 = \int_a^c |g|(x)dx$ and $A_1^2 = (c-a)\int_a^c f^+(x)dx$, this claim is the statement that $A_1^2 > \exp(-A_0)$ implies $A_1^2 + A_0 > 1$. This follows from $e^{-x} + x \geqq 1$ for all $x \geqq 0$.

Now define $B_1^2 = (b-c)\int_c^b f^+(x)dx$ and $B_0 = \int_c^b |g|(x)dx$. Then we have $B_1^2 > \exp(-B_0)$. Now the inequality that is related to (4) as (7) is to (5) is gotten from $4\exp(-y/2) \geqq -2y+4$, $y \geqq 0$.

PROOF OF THE THEOREM. First we have that

$$\int_a^b f^+(x)dx = \frac{A_1^2}{c-a} + \frac{B_1^2}{b-c} \geqq \frac{(A_1+B_1)^2}{b-a}$$

by elementary calculus. In fact, the right member is the minimum of the middle member as function of c. Thus the left-hand side of (3) is greater than

$$\begin{aligned}(A_1+B_1)^2 - 4\exp[-\tfrac{1}{2}(A_0+B_0)] & \\ &\geqq A_1^2 + B_1^2 + 2A_1B_1 - 4\exp[-\tfrac{1}{2}(A_0+B_0)] \\ &> \exp(-A_0) + \exp(-B_0) - 2\exp[-\tfrac{1}{2}(A_0+B_0)] \\ &= [\exp(-\tfrac{1}{2}A_0) - \exp(-\tfrac{1}{2}B_0)]^2 \geqq 0.\end{aligned}$$

This proves (3). Equation (4) follows from the inequality $4\exp(-y/2) + 2y - 4 \geqq 0$ for all $y \geqq 0$, where y is replaced by $\int_a^b |g|$.

We remark that both inequalities (7) and (3) are more enlightening than their counterparts (5) and (4). In particular, they show that $x = (b-a)\int_a^b f^+$ cannot be small unless $y = \int_a^b |g|$ is very large. In fact, as $x \to 0$, $y \to \infty$. This does not follow from (4). Finally, the inequality (3) is sharp since it reduces to Lyaponov's inequality when $g \equiv 0$, and this is known to be sharp, see [3].

ADDED IN PROOF. Professor P. Hartman has pointed out that (3) is announced in Levin, *On linear second order differential equations*, Soviet Math. Dokl. 4 (1963), 1814–1817. The content of this paper is an elementary proof of (3) and the observation that (4) follows from (3).

REFERENCES

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IOWA STATE UNIVERSITY