A NOTE ON DERIVATION PAIRS

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1. **Introduction.** Let G be a region in the complex plane and $H(G)$ denote the vector space of functions analytic on G. Let L and M be two linear functionals on $H(G)$. The pair $\{L, M\}$ is a derivation pair if

$$L(fg) = L(f)M(g) + L(g)M(f), \qquad f, g \in H(G). \tag{1}$$

The purpose of this paper is to determine all derivation pairs generalising a result of L. A. Rubel [1]. This incidentally answers a question raised by him viz., whether the functionals satisfying (1) are continuous.

We denote by I the identity function and I^2 will then denote the function defined by $I^2(z) = I(z)^2$. Throughout we assume $\{L, M\}$ to be a derivation pair and $L \neq 0$.

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THEOREM. *Let $\{L, M\}$ be a derivation pair. Then one of the following is true*:

(i) *there exists a $z_1 \in G$ such that*

$$L(f) = L(1)f(z_1), \quad M(f) = \tfrac{1}{2}f(z_1), \qquad f \in H(G);$$

(ii) *there exists a $z_1 \in G$ such that*

$$L(f) = L(I)f'(z_1), \quad M(f) = f(z_1), \qquad f \in H(G);$$

(iii) *there exists $z_1, z_2 \in G$ $(z_1 \neq z_2)$ such that*

$$L(f) = \frac{L(I)}{z_1 - z_2}(f(z_1) - f(z_2)), \quad M(f) = \tfrac{1}{2}(f(z_1) + f(z_2)), \qquad f \in H(G).$$

2. **Lemmas.**

LEMMA 1. *If N is a multiplicative linear functional on $H(G)$, then there exists a $z_0 \in G$ such that $N(f) = f(z_0)$, $f \in H(G)$.*

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The proof of the above lemma is simple and we include it for the sake of completeness.

PROOF. Since N is multiplicative we have $N(f)=N(f)N(1)$ which implies $N(1)=1$. Let $N(I)=z_0$ so that $N(I-z_0)=N(I)-z_0N(1)=0$. We now claim that this z_0 will satisfy our requirement. First we show that $z_0\in G$. Suppose not. Then $1/(I-z_0)\in H(G)$ and

$$1 = N\left((I - z_0)\frac{1}{(I - z_0)}\right) = N(I - z_0)N\left(\frac{1}{I - z_0}\right) = 0$$

which is impossible.

Now let $f\in H(G)$. Consider the analytic function g defined by $(I-z_0)g=f-f(z_0)$. Applying N to the function $(I-z_0)g$ we obtain $0=N(f)-N(f(z_0))$ or $N(f)=f(z_0)$. This completes the proof of lemma.

LEMMA 2. *If $L(1)\neq 0$, then $L/L(1)$ and $2M$ are multiplicative.*

PROOF. By (1) it follows that $L(1)=2L(1)M(1)$ so that $M(1)=\frac{1}{2}$. Then again using (1) it is easy to show that

$$M(f) = L(f)/2L(1), \qquad f \in H(G).$$

Substituting this value of M in (1) the result follows.

When $L(1)=0$, we have $L(f)=L(f)M(1)$ and since $L\neq 0$, this implies that $M(1)=1$.

LEMMA 3. *Let $\{L, M\}$ be a derivation pair and suppose $L(1)=0$. Then*

(a) $L(I)\neq 0$.

(b) *M is multiplicative when $M(I^2)=M(I)^2$.*

(c) *If f is defined at z_0 and $f\in H(G)$, then*

$$L(f) = L(I)M\left(\frac{f - f(z_0)}{I - z_0}\right), \qquad f \in H(G),$$

where $z_0=M(I)$.

PROOF. (a) Suppose $L(I)=0$. Let $M(I)=z_0$. If $z_0\notin G$, and $f\in H(G)$, define $f(z_0)=0$. For all $f\in H(G)$, $(f-f(z_0))/(I-z_0)\in H(G)$ so that

$$L(f) = L(f - f(z_0)) = L\left((I - z_0)\frac{f - f(z_0)}{I - z_0}\right)$$

$$= L(I)M\left(\frac{f - f(z_0)}{I - z_0}\right) = 0, \qquad f \in H(G).$$

Hence $L=0$ which is a contradiction. This proves (a).

(b) Using (1) we obtain on the one hand

$$L(I^2f) = L(I^2)M(f) + L(f)M(I^2) = 2L(I)M(I)M(f) + L(f)M(I)^2$$

and on the other,

$$\begin{aligned} L(I^2f) &= L(I\cdot If) = L(I)M(If) + L(If)M(I) \\ &= L(I)M(If) + L(I)M(I)M(f) + L(f)M(I)^2. \end{aligned}$$

Comparing the two expressions for $L(I^2f)$ and noting that $L(I)\neq 0$, we get $M(If) = M(I)M(f)$, $f\in H(G)$. From this relation it is easy to show (as in Lemma 1) that there exists a $z_0 \in G$ such that $M(f) = f(z_0)$, $f\in H(G)$. This implies M is multiplicative.

(c) $(f-f(z_0))/(I-z_0)\in H(G)$ and

$$\begin{aligned} L(f) = L(f - f(z_0)) &= L\left((I - z_0)\frac{f - f(z_0)}{I - z_0}\right) \\ &= L(I - z_0)M\left(\frac{f - f(z_0)}{I - z_0}\right) + M(I - z_0)L\left(\frac{f - f(z_0)}{I - z_0}\right) \\ &= L(I)M\left(\frac{f - f(z_0)}{I - z_0}\right) \end{aligned}$$

since $L(1) = 0$ and $M(I-z_0) = 0$.

This completes the proof of lemma.

Now if f, $g\in H(G)$ and are defined at z_0, then applying (c) to $L(fg)$, $L(f)$ and $L(g)$ and substituting in (1) we get

$$M\left(\frac{fg - f(z_0)g(z_0)}{I - z_0}\right) = M(f)M\left(\frac{g - g(z_0)}{I - z_0}\right) + M(g)M\left(\frac{f - f(z_0)}{I - z_0}\right). \tag{2}$$

Put $f = I^2 - z_0 I$,

$$g = 1/(I - z_1) \qquad (z_1 \notin G,\ z_1 \neq z_0).$$

On noting that $I/(I-z_1) = 1 + z_1/(I-z_1)$, (2) simplifies with these special values to

$$M\left(\frac{1}{I - z_1}\right)\left\{\frac{M(I^2) - z_0^2}{z_1 - z_0} + z_0 - z_1\right\} = 1. \tag{3}$$

3. **Proof of theorem.** *Case* (i). $L(1)\neq 0$. Then $L/L(1)$ is multiplicative by Lemma 2. By Lemma 1, there exists $z_1 \in G$ such that $L(f)/L(1)$

$=f(z_1)$, $f\in H(G)$. Also $M(f)=L(f)/2L(1)=f(z_1)/2$, $f\in H(G)$. This gives (i) of our theorem.

Case (ii). $L(1)=0$. We have two possibilities $M(I^2)=M(I)^2$ or $M(I^2)\neq M(I)^2$.

If $M(I^2)=M(I)^2$, then M is multiplicative by Lemma 3. Apply Lemma 1 to get $z_1\in G$ such that $M(f)=f(z_1)$, $f\in H(G)$. Now we will prove $L(f)=L(I)f'(z_1)$. Recall that $M(I)=z_1$. Since $z_1\in G$, it follows that $(f-f(z_1))/(I-z_1)\in H(G)$ for all $f\in H(G)$ and then, by (1) and $L(1)=0$,

$$\begin{aligned}L(f) = L(f-f(z_1)) &= L((I-z_1)(f-f(z_1))/(I-z_1))\\ &= L(I)M((f-f(z_1))/(I-z_1)) = L(I)f'(z_1)\end{aligned}$$

and we obtain (ii) of our theorem.

It remains to consider the case when $M(I^2)\neq M(I)^2=z_0^2$ and $L(I)=0$.

Since $M(I^2)-z_0^2\neq 0$, there are two distinct roots, z_1 and z_2 say, of

$$(M(I^2)-z_0^2)/(z-z_0)+z_0-z=0.$$

These roots satisfy

$$z_1+z_2=2z_0, \qquad z_1z_2=2z_0^2-M(I^2),$$

so that

$$M\{(I-z_1)(I-z_2)\} = M(I^2)-(z_1+z_2)M(I)+z_1z_2M(1)=0. \tag{4}$$

Also by (1), (4) and $L(1)=0$,

$$L\{(I-z_1)(I-z_2)\} = L(I)M(I-z_2)+L(I)M(I-z_1)=0. \tag{5}$$

$z_1\in G$, since otherwise $M(1/(I-z_1))$ would be finite, contradicting (3). Similarly $z_2\in G$.

We can now prove that we have case (iii) of the theorem.

Let $f\in H(G)$. Then

$$g=\frac{1}{I-z_2}\left[\frac{f-f(z_1)}{I-z_1}-\frac{f(z_2)-f(z_1)}{z_2-z_1}\right]\in H(G).$$

Applying (1), (4), and (5) to

$$(I-z_1)(I-z_2)g=\frac{1}{z_2-z_1}[(z_2-z_1)(f-f(z_1))-(I-z_1)(f(z_2)-f(z_1))].$$

We obtain

$$L(f) = \frac{L(I)}{z_1 - z_2}\,[f(z_1) - f(z_2)], \qquad f \in H(G).$$

From the relation $L((I-z_0)f) = L(I)M(f)$, $f \in H(G)$, we obtain $M(f) = \frac{1}{2}(f(z_1)+f(z_2))$, $f \in H(G)$. The theorem is completely proved.

Reference

1. L. A. Rubel, *Derivation pairs on the holomorphic functions*, Funkcial. Ekvac. **10** (1967), 225–227.

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