A CHARACTERIZATION OF CERTAIN CONFORMALLY EUCLIDEAN SPACES OF CLASS ONE

G. M. LANCASTER

1. In this paper we will examine the metrics of conformally Euclidean spaces C_n $(n \geqq 4)$ having the following two properties:

(1) They are locally and isometrically imbeddable in Euclidean space of one higher dimension (E_{n+1}), i.e. they are of class one.

(2) With respect to a conformal coordinate system, the matrix of the second fundamental tensor $[b_{ij}]$ has diagonal form.

The condition for class one is that there exist a (second fundamental) tensor $[b_{ij}]$, satisfying the Gauss (1.1) and Codazzi (1.2) equations:

$$R_{hijk} = b_{hj}b_{ik} - b_{hk}b_{ij}, \tag{1.1}$$

$$b_{ij,k} = b_{ik,j}. \tag{1.2}$$

To satisfy (2), we will therefore look for a solution of these equations for which $b_{ij}=0$ when $i \neq j$.

Sen, in a series of papers ([4], [5] and [6]), has investigated certain conditions for a C_n to be of class one, and obtained [6, Theorem 3] a canonical form for the metric of such a space. His result, however, is incorrect in its full generality (see [3] for a disproof). In 1962, at the meeting of the International Congress of Mathematicians in Stockholm [8], R. Blum presented, without proof, a canonical form for the metric of a C_n satisfying (1) and (2) above and such that $n \geqq 4$. In his theorem, however, Blum overlooked an exception, and it is therefore not correct as stated. It is the purpose of this paper to give a proof and a simplification of the corrected result.

Thomas [7] showed that when r, the rank of the matrix $[b_{ij}]$, is greater than or equal to four, equations (1.2) follow as a consequence of equations (1.1). It is therefore logical to consider separately the cases $n \geqq 4$ and $n=3$ (the case $n=2$ is not considered here; the surfaces $\overline{C}_2^1$ are called isothermal surfaces and form a separate area of study in themselves). It will turn out that for $n \geqq 4$, r is greater than or equal to four except in two particular cases. In both, however, it is easily verified that equations (1.2) are satisfied because of (1.1). For $n=3$, the situation is somewhat different and the Codazzi equations must be considered separately as a set of independent conditions. This case will be the object of investigation in a later paper.

Received by the editors August 17, 1967.

As regards notation, a C_n having property (1) will be denoted C_n^1, following Sen's example, while a C_n which has both properties (1) and (2) will be denoted $\overline{C}_n^1$. Tensor notation used throughout will be essentially that to be found in Eisenhart [2].

2. Referred to a conformal coordinate system, the metric of a C_n is

$$ds^2 = e^{2\sigma} \sum_i (dx^i)^2, \tag{2.1}$$

where $\sigma = \sigma(x^1, x^2, \cdots, x^n)$ and $e^{-2\sigma} \neq 0$.

By routine calculation we then obtain the following expression for the Riemann Curvature Tensor:

$$R_{hijk} = e^{2\sigma}\left[\delta_{hk}\sigma_{ij} + \delta_{ij}\sigma_{hk} - \delta_{hj}\sigma_{ik} - \delta_{ik}\sigma_{hj} + \sum_m \sigma_{,m}^2(\delta_{hj}\delta_{ik} - \delta_{hk}\delta_{ij})\right]$$

where $\sigma_{ij} = \sigma_{,ij} + \sigma_{,i}\sigma_{,j}$, and $\sigma_{,ij}$ (the second covariant derivative of σ), is given by

$$\sigma_{,ij} = \partial_i\partial_j\sigma - 2\sigma_{,i}\sigma_{,j} + \delta_{ij}\sum_m \sigma_{,m}^2.$$

Substituting this expression into (1.1) and considering components yields the following two equations:

$$\sigma_{ij} = 0 \qquad (i \neq j; j = 1, \cdots, n), \tag{2.2}$$

and

$$b_{hh}b_{ii} = e^{2\sigma}\left(\sum_m \sigma_{,m}^2 - \sigma_{hh} - \sigma_{ii}\right) \qquad (h \neq i;\ h, i = 1, \cdots, n). \tag{2.3}$$

We will now consider each of these relations in turn.

3. Equation (2.2) simplifies to

$$\partial_i\partial_j\sigma - \partial_i\sigma\partial_j\sigma = 0 \qquad (i \neq j).$$

If we now multiply this by $e^{-\sigma}$ we obtain

$$\partial_i\partial_j e^{-\sigma} = 0 \qquad (i \neq j).$$

Thus

$$\partial_j e^{-\sigma} = F(x^j) \qquad (j = 1, 2, \cdots, n),$$

and hence

$$e^{-\sigma} = \sum_m f_m \tag{3.1}$$

where f_m is a function of x^m only.

4. Utilizing equation (3.1), (2.3) reduces to

$$b_{hh}b_{kk} = e^{4\sigma}[e^{-\sigma}(f_h'' + f_k'') - A] \qquad (h \neq k), \tag{4.1}$$

where

$$A = \sum_m f_m'^2. \tag{4.2}$$

Similarly

$$b_{ii}b_{jj} = e^{4\sigma}[e^{-\sigma}(f_i'' + f_j'') - A] \qquad (i \neq j). \tag{4.3}$$

Multiplying (4.1) and (4.3) we obtain

$$\begin{aligned} b_{hh}b_{kk}b_{ii}b_{jj} = e^{8\sigma}[&e^{-2\sigma}(f_h'' + f_k'')(f_i'' + f_j'') \\ &- A\, e^{-\sigma}(f_h'' + f_k'' + f_i'' + f_j'') + A^2] \\ &(h \neq k,\ i \neq j), \end{aligned} \tag{4.4}$$

and similarly:

$$\begin{aligned} b_{hh}b_{ii}b_{kk}b_{jj} = e^{8\sigma}[&e^{-2\sigma}(f_h'' + f_i'')(f_k'' + f_j'') \\ &- Ae^{-\sigma}(f_h'' + f_i'' + f_k'' + f_j'') + A^2] \\ &(h \neq i,\ k \neq j). \end{aligned} \tag{4.5}$$

Equate (4.4) and (4.5) and simplify. Then

$$(f_i'' - f_k'')(f_j'' - f_h'') = 0 \qquad (i \neq h,\ i \neq j,\ k \neq j,\ k \neq h).$$

From this expression, we then deduce the result that $f_i'' = 2a$ (constant) for all i except one value, say $i = 1$. Thus

$$f_i = ax^{i^2} + b_i x^i + c_i \qquad (i = 2, 3, \cdots, n) \tag{4.6}$$

while f_1 is arbitrary.

Putting $f = f_1 + \sum_{i=2}^n c_i$ and substituting (4.6) and (3.1) into (2.1) we thus obtain

$$ds^2 = \frac{\sum_{i=1}^n (dx^i)^2}{\left[f(x^1) + a\sum_{m=2}^n (x^m)^2 + \sum_{m=2}^n b_m x^m\right]^2}. \tag{4.7}$$

It is then a fairly straightforward matter to obtain explicit expressions for the b_{ii} from equation (4.3), viz

$$b_{ii} = e^{2\sigma}\left(4af - f'^2 - \sum_{m=2}^n b_m^2\right)^{1/2} \qquad (i = 2, 3, \cdots, n), \tag{4.8}$$

and

$$
\begin{aligned}
(4.9)\qquad b_{11} &= b_{ii} + \frac{e^{3\sigma}(f'' - 2a)}{b_{ii}} \qquad \text{if } b_{ii} \neq 0 \quad (i \neq 1),\\
&= 0 \qquad \text{if } b_{ii} = 0 \quad (i \neq 1).
\end{aligned}
$$

However, there is an interesting exception which arises when $b_{ii}=0$ $(i=2, \cdots, n)$ and $a\neq 0$. If we equate equation (4.8) to zero and solve, we obtain the following two independent solutions:

$$
(1)\qquad f(\overset{1}{x}) = a\overset{1}{x}{}^{2} + b_1\overset{1}{x} + \sum_{m=1}^{n} \overset{2}{b}_m/4a,
$$

$$
(2)\qquad f(\overset{1}{x}) = \sum_{m=2}^{n} \overset{2}{b}_m/4a.
$$

Solution (1) implies that $\overline{C}_n^1$ is a Euclidean space and hence $b_{11}=0$ also, as indicated in equation (4.9).

Solution (2), however, yields a contradiction to the effective Gauss equations (4.1), and hence represents a space which is not a $\overline{C}_n^1$. We may see this by direct substitution into equations (4.1);

$$
\begin{aligned}
h, i \neq 1 \quad &\text{yields} \quad 4ae^{-\sigma} - A = 0,\\
h = 1, i \neq 1 \quad &\text{yields} \quad 2ae^{-\sigma} - A = 0,
\end{aligned}
$$

and together these imply $e^{-\sigma}=0$, i.e. a contradiction.

Furthermore, the space C_n, whose metric is given by equation (4.7) with $f(x^1)=\sum_{m=2}^{n} b_m^2/4a$, is not even of class one, i.e. a C_n^1. This may be seen by obtaining the components of the curvature tensor from this metric and looking for a solution $[b_{ij}]$, not necessarily diagonal, to equations (1.1). A contradiction is thereby obtained. The space is in fact of class two (see [1]).

Thus to equation (4.7) we must add the condition that $f(x^1) \neq \sum_{m=2}^{n} b_m^2/4a$.

5. The Codazzi equations follow because of Thomas' result, except for the following situations:

(a) $b_{11}=b_{ii}=0$ $(i=2, \cdots, n)$, in which case they are satisfied identically.

(b) $b_{11}=0$, $b_{ii}\neq 0$ $(i=2, \cdots, n)$, and $n=4$. This situation occurs when $b_{ii}^2=e^{3\sigma}(2a-f'')$ $(i=2, 3, 4)$ $(f''\neq 2a)$, and f satisfies the differential equation

$$
f(f'' + 2a) - f'^{2} + (f'' - 2a) \sum_{m=2}^{4} (ax^{m^2} + b_m x^m) - \sum_{m=2}^{4} b_m^2 = 0.
$$

It can be fairly readily verified that here again the Codazzi equations are satisfied (by converting these equations to the simpler form

$$\partial_k b_{ii} = \sigma_{,k}(b_{ii} + b_{kk}) \qquad (i \neq k)$$

and checking all the cases).

6. Conversely, if we are given a C_n with metric (4.7), and $f(x^1) \neq \sum_{m=2}^n b_m^2/4a$, we may construct a tensor $[b_{ij}]$ using equations (4.8) and (4.9) such that $b_{ij}=0$ for $i\neq j$. These in turn satisfy the Gauss and Codazzi equations. Furthermore, the tensor $[b_{ij}]$ is unique except for sign provided that rank $[b_{ij}]$ $(=\tau)\geqq 3$ (see [7, p. 188]). This is always true unless C_n is a Euclidean space (in which case it is of class zero anyway).

7. We can further simplify the metric (4.7) by considering separately the cases when $a=0$ and $a\neq 0$.

$a\neq 0$. The transformation $y^1=ax^1$, $y^m=ax^m+b_m/2$ $(m=2, 3, \cdots, n)$ changes the metric to the simpler form

$$(7.1)\quad ds^2 = \frac{\sum_{i=1}^n (dy^i)^2}{[F(y^1)+\theta]^2} \quad \text{where} \quad \theta = \sum_{i=2}^n (y^i)^2 \quad \text{and} \quad F(y^1) \neq 0.$$

$a=0$. Here if b_m $(m=2, 3, \cdots, n)$ are all zero, we obtain the metric

$$ds^2 = \frac{\sum_{i=1}^n (dx^i)^2}{[f(x^1)]^2},$$

whereas if the b_m are not all zero, we may make any orthogonal transformation such that

$$y^1 = x^1, \qquad y^2 = \sum_{m=2}^n \frac{b_m}{B} x^m, \quad \text{where} \quad B = (b_2^2 + b_3^2 + \cdots + b_n^2)^{1/2},$$

and obtain the metric

$$ds^2 = \frac{\sum_{i=1}^n (dy^i)^2}{[f(y^1)+By^2]^2}.$$

Thus in both cases when $a=0$, the metric of a $\overline{C}_n^1$ reduces to the form

$$ds^2 = \frac{\sum_{i=1}^{n} (dx^i)^2}{[f(x^1) + Kx^2]^2} \tag{7.2}$$

where K is an arbitrary constant.

8. The following theorem summarizes the results obtained in the preceding sections:

THEOREM. *Let $\overline{C}_n^1$ $(n \geqq 4)$ be a conformally Euclidean space of class one, such that, with respect to a conformal coordinate system $x^1, x^2, \cdots, x^n$, the second fundamental tensor has diagonal form. Then the metric of $\overline{C}_n^1$ takes one of the following two distinct canonical forms:*

$$\text{(I)} \qquad ds^2 = \frac{\sum_{i=1}^{n} (dx^i)^2}{[f(x^1) + \theta]^2} \quad \textit{where} \quad \theta = \sum_{i=2}^{n} (x^i)^2,$$

$$\text{(II)} \qquad ds^2 = \frac{\sum_{i=1}^{n} (dx^i)^2}{[g(x^1) + Kx^2]^2},$$

where f and g are arbitrary twice differentiable functions of x^1 only, except that $f(x^1) \neq 0$, and K is an arbitrary constant.

Conversely, if a C_n $(n \geqq 4)$ possesses either of the metrics (I) *or* (II), *then it is a $\overline{C}_n^1$.*

REFERENCES

1. H. W. Brinkman, *On Riemannian spaces conformal to Euclidean spaces*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 1–3.

2. L. P. Eisenhart, *Riemannian geometry*, 5th printing, Princeton Univ. Press, Princeton, N. J., 1964.

3. G. M. Lancaster, *Note on a theorem by Sen*, Proc. Amer. Math. Soc. **19** (1968), 1119–1122.

4. R. N. Sen, *On a type of Riemannian space conformal to a flat space*, J. Indian Math. Soc. **21** (1957), 105–114.

5. ———, *Conformally Euclidean space of class one*, Indian J. Math. **6** (1964), 93–103.

6. ———, *On conformally-flat Riemannian space of class one*, Proc. Amer. Math. Soc. **17** (1966), 880–883.

7. T. Y. Thomas, *Riemannian spaces of class one and their characterization*, Acta Math. **67** (1936), 169–211.

8. International Congress of Mathematicians, Abstract of short communications, Stockholm 1962, Almquist and Wiksells, Uppsala, 1962.

UNIVERSITY OF SASKATCHEWAN