SOME REMARKS ON HYPERSPACES

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1. The purpose of this paper is to answer a question of R. Schori [**3**] and to provide simpler arguments for some generalizations of Schori's results.

If X is a metric space, the *hyperspace* of X, denoted 2^X, is the space of all nonvoid closed subsets of X with the usual Hausdorff metric. The *n-fold* ($n \geqq 1$) symmetric product (Borsuk-Ulam [**1**]) of X, denoted $X(n)$, is the subspace of 2^X consisting of all elements with $\leqq n$ points. Let I denote the closed unit interval, I^n the n-cube and I^∞ the Hilbert cube. Let $S(X)$ denote the subspace of 2^X consisting of all continua. In [**3**] R. Schori shows that for $n \geqq 1$ and $\alpha = \infty, 1, 2, \cdots, I^\alpha(n)$ contains I^α as a factor; that is, $I^\alpha(n)$ is homeomorphic to $Y \times I^\alpha$ for some space Y. Let J^∞ denote another copy of the Hilbert cube with $J = [-1, 1]$ and let R be the equivalence relation on J^∞ defined by identifying each $x = (x_1, x_2, \cdots)$ with $-x = (-x_1, -x_2, \cdots)$.

THEOREM I. *J^∞/R is not homeomorphic to J^∞.*

Thus we settle a question of R. Schori [**3**].

PROOF. Let us suppose it were. Consider the natural quotient map $P: J^\infty \to J^\infty/R$. Evidently the restriction of $P: J^\infty - 0 \to J^\infty/R - P(0)$ is a two-fold covering. Since the Hilbert cube is homogeneous, it follows from the assumption that $J^\infty/R - P(0)$ is simply connected and therefore (well-known) does not admit a two-fold covering. This is a contradiction.

Question. Is J^∞/R an Absolute Retract?

The question is interesting because J^∞/R is clearly a retract of $J^\infty/R \times J^\infty$, which is homeomorphic to $J^\infty(2)$ by [**3**]. A negative answer would imply that $J^\infty(2)$ is not homeomorphic to J^∞.

THEOREM II. *Let m, n be positive integers. If $X = I^m(n)$, $2^{(I^m)}$ or $S(I^m)$, then X contains I^m as a factor.*

REMARK. Schori's proof is restricted to symmetric products since it makes strong use of the following well-known characterization of $I^m(n)$. If n is a positive integer, then $I^m(n)$ is homeomorphic to I^m/R where R is the equivalent relation on I^m defined by $(x_1, \cdots, x_n)$

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$R(y_1, \cdots, y_n)$ iff $\{x_1, \cdots, x_n\} = \{y_1, \cdots, y_n\}$ (x_i and y_i are points in I^m). However, as such he is able to include the case when $m=\infty$ in his theorem. On the other hand, by working directly with the subspaces of 2^{I^m} we are able to give a much simplified proof and although we are not able to include $m=\infty$, we generalize in the direction of more general subspaces of 2^{I^m} which, as the nature of the technique, may include even more subclasses than those mentioned in Theorem II. In the case when $m=\infty$ we are able to prove the following partial generalization:

THEOREM III. *If $X = I^\infty(n)$, $S(I^\infty)$ or 2^{I^∞}, then for any positive integer k, X contains I^k as a factor.*

Question. If $X = S(I^\infty)$ or 2^{I^∞}, must X contain I^∞ as a factor?

2. **The Cone Lemma.** The *cone* over a space X, denoted $C(X)$, is the quotient space of $X \times I$ obtained by identifying $X \times 1$ as a point v, where v is called the vertex of $C(X)$. Inductively for $n > 1$, define $C^n(X) = C(C^{n-1}(X))$. Let "$\approx$" denote "homeomorphic to".

LEMMA. (SCHORI). *For $n > 1$, $C^n(X) \approx C(X) \times I^{n-1}$.*

OUTLINE OF PROOF. By induction it suffices to consider $n=2$; that is $C^2(X) \approx C(X) \times I$. For each $x \in X$, $C^2(x)$ can be realized as a triangle and thus we can deform $C^2(x)$ into $C(x) \times I$. If we do this uniformly for each x (detail in [**3**]), we obtain a homeomorphism from $C^2(X)$ onto $C(X) \times I$.

PROOF OF THEOREM II. Let $\{v_i\}$ be the unit points in Euclidean space E^{m+1}; that is, v_i has 1 for its ith-coordinate and 0 otherwise. Let σ denote the m-simplex $v_1 v_2 \cdots v_{m+1}$. Since $\sigma \approx I^m$, it is clear we can replace I^m by σ in Theorem II. For each i let σ_i be the $(m-1)$-dimensional face $v_1 \cdots \hat{v}_i \cdots v_{m+1}$. Now let X_0 be any space in Theorem II. For $i \geqq 1$ let $X_i = \{x \in X_0 \mid x \cap \sigma_k \neq \emptyset \text{ for all } k \leqq i\}$. Clearly $C(X_{i+1}) \approx \{tv_{i+1} + (1-t)x \colon t \in I, x \in X_{i+1}\}_1 \subset X_i$. We contend that $\{tv_{i+1} + (1-t)x \colon t \in I, \ x \in X_{i+1}\} = X_i$. Suppose $x (\neq v_{i+1}) \in X_i$. Let $t = \min \pi_{i+1}(x)$, where π_{i+1} is the usual projection map. It is routine to verify that $x' = x/(1-t) - tv_{i+1}/ \in X_{i+1}$ and thus $tv_{i+1} + (1-t)x' = x$. Inductively, we have $X_0 \approx C(X_1) \approx CC(X_2) \cdots \approx C^{m+1}(X_{m+1})$. The theorem now follows from the Cone Lemma.

PROOF OF THEOREM III. Let s denote the infinite product of reals and let $T = \{(x_1, x_2, \cdots) \in I^\infty \mid 0 \leqq x_i \leqq 1 \text{ and } \sum_{i=1}^\infty x_i \leqq 1\}$. Evidently T is closed in I^∞ and therefore compact. Thus T is a compact convex subset of the locally convex topological linear space s which admits a countable family of continuous linear forms that separate points (namely, the family $\{\pi_i\}$ of projections) and thus by [2] T is homeomorphic to I^∞. Hence we may replace I^∞ by T in Theorem III. Now let X_0 be any space in Theorem III and let k be any positive integer. For each $i \geqq 1$, let $T_i = \{(x_1, x_2, \cdots) \in T \mid x_i = 0\}$ and $X_i = \{x \in X_0 \mid x \cap T_k \neq \emptyset \text{ for all } k \leqq i\}$. As in Theorem II, $C(X_{i+1}) \approx X_i$. Inductively, $X_0 \approx C(X_1) \cdots \approx C^{k+1}(X_{k+1})$. The theorem now follows from the Cone Lemma.

References

1. K. Borsuk and S. Ulam, *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. **37** (1931), 875–882.

2. V. L. Klee, Jr., *Some topological properties of convex sets*, Trans. Amer. Math. Soc. **78** (1955), 30–45.

3. R. Schori, *Hyperspaces and symmetric products of topological spaces*, Fund. Math. **63** (1968), 77–88.

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