# SEMI-ISOMORPHISMS OF CERTAIN INFINITE PERMUTATION GROUPS

W. R. SCOTT

Let $X$ and $Y$ be infinite cardinal numbers, $S(X)$ the full symmetric group on a set of cardinal $X$, $A(X)$ the alternating group of finite even permutations on the same set, and $S(X, Y)$ the subgroup of $S(X)$ of all permutations moving fewer than $Y$ elements.

A semi-automorphism of a group $G$ is a permutation $T$ of $G$ such that $(xyx)T=(xT)(yT)(xT)$ for all $x, y\in G$. Semi-isomorphism is defined similarly. Dinkines [1] and Herstein and Ruchte [2] showed that any semi-automorphism of $S(X, Y)$ or $A(X)$ was either the restriction $T$ of an inner automorphism of $S(X)$, or was of the form $T(-I)$ where $x(-I)=x^{-1}$ for all $x$. Theorem 11.4.6 of [3] states that every automorphism of any group $G$ such that $A(X)\subset G\subset S(X)$ is the restriction of an inner automorphism of $S(X)$. In the present paper, we prove the common generalization of these two theorems whose statement is obvious. (See the corollary at the end.)

LEMMA. *If $Q$ is a subset of $A(X)$ containing all 3-cycles and such that $x, y\in Q$ imply $xyx\in Q$, then $Q=A(X)$.*

PROOF.[1] If $x\in A(X)$, then $x=c_1c_2 \cdots c_n$, where the $c_i$ are 3-cycles. If $n=1$, then $x\in Q$. Induct on $n$. Since

$$x = c_1 \cdots c_n = c_1^{-1}(c_1^{-1}(c_2 \cdots c_n)c_1)c_1^{-1}$$

and the middle factor is the product of $n-1$ 3-cycles $c_1^{-1}c_ic_1$, it follows by induction that $x\in Q$.

THEOREM. *Let $X$ be an infinite cardinal number, $G$ and $H$ subgroups of $S(X)$ containing $A(X)$, and $T$ a semi-isomorphism of $G$ onto $H$. Then either*

(1) *$T$ is induced by conjugation by an element of $S(X)$, or*

(2) *$T$ is the product of a mapping of type* (1) *mapping $G$ onto $H$, and the mapping $-I$ of $H$ onto $H$.*

PROOF. Since the center of $H$ is 1, it follows that $T$ preserves order and powers [1, Lemma 1]. Let $S$ ($S'$) be the subgroup generated by all elements of order 2 in $G$ ($H$). Since $A(X)$ is simple, $S$ and $S'$ each contain $A(X)$. Call elements $x, y\in G$, $S$-conjugate iff $x=y^s$ for some

Received by the editors August 16, 1968.

[1] This brief proof is due to Fletcher Gross.

$s \in S$. Define $S'$-conjugacy similarly. Then $S$- and $S'$-conjugacy are equivalence relations. Moreover, if $x$ is $S$-conjugate to $y$, then $x = u_n \cdots u_1 y u_1 \cdots u_n$ with $o(u_i) = 2$, so that

$$xT = (u_n T) \cdots (u_1 T)(yT)(u_1 T) \cdots (u_n T),$$

$o(u_i T) = 2$, and $xT$ and $yT$ are $S'$-conjugate. Since the inverse of a semi-isomorphism is also a semi-isomorphism, the converse is also true. Therefore $T$ carries an $S$-conjugate class onto an $S'$-conjugate class.

Let $M$ be the set of 3-cycles. Then

(3) $M$ is an $S$-conjugate class of $G$.

(4) All elements of $M$ have order 3.

(5) Max $o(xyx) = 5$ for $x, y \in M$.

By earlier remarks, $MT$ satisfies these conditions with $H$ instead of $G$ and $S'$ instead of $S$. If $x = (123)(456) \in MT$, then conjugation by an appropriate $s \in A(X) \subset S'$ gives $y = (132)(478) \in MT$. Then

$$xyx = (123)(46785)$$

contrary to (5). If $x = (123)(456)(789) \cdots \in MT$, then conjugation by some $s \in S'$ yields $y = (132)(457)(689) \cdots \in MT$. But then

$$xyx = (123)(48)(59) \cdots,$$

contrary to (5). By (4), it follows that all elements of $MT$ are 3-cycles. Hence, by (3), $MT = M$.

Let $Q = \{x \in A(X) \mid xT \in A(X)\}$. If $x, y \in Q$, then $xyx \in Q$. By the lemma, $Q = A(X)$, that is $A(X)T \subset A(X)$. Using $T^{-1}$ instead of $T$, we have $A(X)T = A(X)$. Thus $T \mid A(X)$ is a semi-automorphism. By [1] or [2], $T \mid A(X)$ is either an automorphism or an anti-automorphism. But all automorphisms of $A(X)$ are of the form $T_z$ where $T_z$ is conjugation by some element $z \in S(X)$ (see, for example, [3, Theorem 11.4.8]). If $T$ is an anti-automorphism on $A(X)$, then $T(-I)$ is an automorphism, hence $T(-I) = T_z$ and $T = T_z(-I)$ on $A(X)$.

Let $U = TT_z^{-1}$ or $T(-I)T_z^{-1}$ in the above two cases. Then $U$ is a semi-isomorphism of $G$ which is the identity on $A(X)$. The theorem will follow if we can show that $U$ is the identity on $G$.

Suppose that there is an $x \in G$ such that $xU \neq x$. We assert that $x$ can be chosen so that it fixes at least 5 letters. If this is false, then choose $x$ so that it fixes the maximum possible number of letters (at most 4). If $x$ contains an $n$-cycle $n \geqq 3$, we can assume that $x = (\cdots 123 \cdots) \cdots$, in which case $(123)x(123)$ fixes all letters fixed by $x$ and the letter 3 in addition. In the other case, $x$ is a product of disjoint 2-cycles, say $x = (12)(34) \cdots$; then $(123)x(123)$ again fixes 3 and all letters fixed by $x$. Since for $y=(123)$, $(yxy)U=y(xU)y \neq yxy$, we have a contradiction in either case. Hence, as asserted, $x$ can be chosen so that it fixes at least 5 letters.

Now let $x$ be any element of $G$ fixing at least 5 letters. We assert that $xU$ fixes the same letters as $x$. Suppose, in fact, that $x$ fixes 1, 2, 3, 4, and 5, but that $xU$ moves 1. Then, changing notation if necessary, $1(xU)\neq 1$, 2, 3, or 4. Now

$$xU = [(12)(34)x(12)(34)]U = (12)(34)(xU)(12)(34),$$
$$(xU)(12)(34) = (12)(34)(xU).$$

But the left side sends 1 into $1(xU)$, while the right sends it into $2(xU)$. This contradiction proves that $xU$ fixes all letters fixed by $x$. Consideration of $U^{-1}$ shows that $xU$ fixes the same letters as $x$.

Let $x$ fix at least 5 letters, and $xU\neq x$. For some letter $i$, $ix=j$, $i(xU)=k$, $k\neq j$. The preceding paragraph implies that $i\neq j$. Let $r\neq i$ or $j$. Now the element $(rij)x(rij)$ fixes $r$ and all letters fixed by $x$, hence at least 5 letters altogether. However $(rij)(xU)(rij)=[(rij)x(rij)]U$ does not fix $r$. This contradicts the preceding paragraph. Thus the theorem is true.

COROLLARY. *If $A(X)\subset G\subset S(X)$ and $T$ is a semi-automorphism of $G$, then there is an element $z\in N_{S(X)}(G)$ such that either $T$ is the automorphism $T_z$ (induced by conjugation by $z$) or the anti-automorphism $T_z(-I)$.*

## BIBLIOGRAPHY

1. Flora Dinkines, *Semi-automorphisms of symmetric and alternating groups*, Proc. Amer. Math. Soc. 2 (1951), 478–486.

2. I. N. Herstein and M. F. Ruchte, *Semi-automorphisms of groups*, Proc. Amer. Math. Soc. 9 (1958), 145–150.

3. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

UNIVERSITY OF UTAH