ON ADDITIVE FUNCTIONALS[1]

N. A. FRIEDMAN AND M. KATZ

Let $C(S)$ denote the class of bounded real-valued continuous functions defined on a topological space S. In [1] a generalization of the Riesz representation theorem is proven for a class of additive functionals defined on $C(S)$ where $S=[0, 1]$. This result was extended to the case where S is a compact metric space in [2]. In this note we show that the result also holds when S is a compact Hausdorff space.

Let $f\in C=C(S)$. We define the norm of f as $\|f\|=\max_{s\in S}|f(s)|$. We shall say a functional ϕ defined on C is an additive functional if ϕ satisfies the following three conditions:

(C1) *Continuity*. For each $\epsilon>0$ and $b>0$ there exists $\delta=\delta(\epsilon, b)>0$ such that $\|f\|\leqq b$, $\|g\|\leqq b$, and $\|f-g\|\leqq\delta$ imply $|\phi(f)-\phi(g)|\leqq\epsilon$.

(C2) *Boundedness*. For each $b>0$ there exists $B=B(b)$ such that $\|f\|\leqq b$ implies $|\phi(f)|\leqq B$.

(C3) *Additivity*. If $f_1f_2\equiv 0$ then $\phi(f_1+f_2+f)=\phi(f_1+f)+\phi(f_2+f)-\phi(f)$. (C3) is the generalization of the measure theory equality

$$m(A_1\cup A_2\cup A) = m(A_1\cup A) + m(A_2\cup A) - m(A),$$

which holds when $A_1\cap A_2=\varnothing$. In particular, if $\phi(0)=0$, then letting $f=0$ in (C3), we obtain $\phi(f_1+f_2)=\phi(f_1)+\phi(f_2)$ when $f_1f_2\equiv 0$. The representation theorem is stated as follows.

THEOREM. *Let S be a compact Hausdorff space. Then ϕ is an additive functional on C if and only if*

$$\phi(f) = \int_S K(f(s), s)\mu(ds),$$

where

(i) *μ is a measure of finite variation defined on the Borel sets of S,*

(ii) *$K(x, s)$ is a measurable function of s for each x,*

(iii) *$K(x, s)$ is a continuous function of x for μ—a.a.s.,*

(iv) *for each $b>0$ there exists $H=H(b)$ such that $|x|\leqq b$ implies $|K(x, s)|\leqq H$ for μ—a.a.s.*

The proof of necessity follows as in [1]. The proof of sufficiency is obtained by carrying over the proof in [2] by utilizing the analogy

Received by the editors June 1, 1967.

[1] Research supported by the National Science Foundation Grant GP 5217.

between the distance function in a metric space and the uniformity which generates the topology on S. We shall first introduce our notation and then give a brief outline of the proof. Since the complete proof is rather long and is essentially unchanged once the appropriate translation into uniformity notation has been made, we shall only prove the initial lemmas in order to indicate the translation.

Let U be the uniformity consisting of neighborhoods of the diagonal Δ in $S\times S$. Therefore if $f\in C$, then f is uniformly continuous with respect to U. We let η and w denote open symmetric neighborhoods of Δ. U is directed by inclusion and we shall denote limits by $\lim_{w\to\Delta}$. Given $a\in S$, let $\eta(a)=\{b: (a, b)\in\eta\}$ and $\eta(A)=\bigcup_{a\in A}\eta(a)$ for $A\subset S$. Note that $\eta(A)$ is always an open set in S. The η-rim about A is defined as $R(A,\eta) = A^c\cap\eta(A)$. We write $g\in R(A,\eta)$ if $g(x) = 0$, $x\notin R(A, \eta)$. Thus $g\in R(A, \eta)$ if g has its support in $R(A, \eta)$. Given h real, we write $g\in R(A, \eta, h)$ if $g\in R(A, \eta)$ and $\|g\|\leqq|h|$. We let $P(A, w, h)$ denote the set of $f\in C$ such that $\|f\|\leqq|h|$; there exists η_f such that $\eta_f\circ\eta_f\subset w$ and $f(s)=h$ for $s\in\mathrm{Cl}(\eta_f(A))$; and $f(s)=0$ for $s\in w(A)^c$. Briefly, $f\in P(A, w, h)$ means f has height h on a closed neighborhood of A and f is zero off of $w(A)$. Given disjoint closed sets F_1 and F_2, we let $u\in C$ denote a function such that $0\leqq u\leqq 1$, $u(F_1)=1$, and $u(F_2)=0$. The existence of u is guaranteed by Urysohn's Lemma. We shall now give an outline of the stages in the proof and shall indicate the lemmas in [2] corresponding to each stage.

It is first shown that $\lim_{w\to\Delta}\phi(f)=\lambda_h(A)$ exists for $f\in P(A, w, h)$ under the assumption $\phi(0)=0$ (Lemmas 1–3). It is then shown that $\lambda_h(F)$, $F\in\mathfrak{F}$, defines a finitely-additive set function on the closed sets $\mathfrak{F}$ (Lemma 4). Utilizing a standard method, we obtain $\lambda_h=\lambda_h^+-\lambda_h^-$ where λ_h^+ and λ_h^- are nonnegative, finitely-additive set functions on $\mathfrak{F}$. It is then shown that λ_h^+ is a regular content on $\mathfrak{F}$ with an identical proof holding for λ_h^- (Lemmas 5–14). Thus λ_h^+ and λ_h^- can be extended to regular measures μ_h^+ and μ_h^- on the Borel sets $\mathfrak{B}$ of S. Letting $\mu_h=\mu_h^+-\mu_h^-$, we obtain a regular signed measure on $\mathfrak{B}$ such that $\mu_h(F)=\lambda_h(F)$, $F\in\mathfrak{F}$. We define a measure $\mu=\sum_{i=1}^{\infty}(\mu_{h_i}^++\mu_{h_i}^-)/2^i\|\mu_{h_i}\|$, where $\{h_i\}$ is a countable dense set of reals and $\|\mu_{h_i}\|$ is the total variation of μ_{h_i}. It follows that $\mu\gg\mu_h$ for each h (Lemma 16). Thus the Radon-Nikodym Theorem yields a kernel $K(h, s)$ which is $\mathfrak{B}$-measurable in s for each h such that

$$\mu_h(B) = \int_S K(h\psi_B(s), s)\mu(ds),$$

where ψ_B is the characteristic function of $B\in\mathfrak{B}$. As in the proofs of

Lemmas 10–12 [1],[2] it can now be shown that there exist a measure μ_* and a kernel $K_*(h, s)$ satisfying (i)–(iv) of the above theorem such that

$$\mu_h(B) = \int_S K_*(h\psi_B(s), s)\mu_*(ds), \qquad B \in \mathcal{B}.$$

Here $K_*(0, s) \equiv 0$ because we have assumed $\phi(0) = 0$. For $f \in C$ we now define

$$\phi_1(f) = \int_S K_*(f(s), s)\mu_*(ds),$$

and show $\phi(f) = \phi_1(f)$ (Lemmas 17, 18). The assumption that $\phi(0) = 0$ is then easily removed.

We shall now prove the existence of $\lim_{w \to \Delta} \phi(f) = \lambda_h(A)$ where $f \in P(A, h, w)$. The proof of Lemma 1 is immediate.

LEMMA 1. *Let $A \subset S$ and $\eta_1 \circ \eta_1 \subset \eta_2$. If $F_1 = \mathrm{Cl}(\eta_1(A))$ and $F_2 = \eta_2(A)^c$ then $F_1 \cap F_2 = \emptyset$.*

LEMMA 2. *Let $f \in C$, $A \subset S$, and $w \in U$. If $g_* \in R(A, w)$ and $\epsilon > 0$ then there exists $g \in C$ and $\eta \in U$ such that g has support in $R(A, \eta)^c \cap R(A, w)$ and $|\phi(f+g) - \phi(f+g_*)| < \epsilon$.*

PROOF. (C1) implies that if $\|g\| \leqq \|g_*\|$ then there exists $\delta > 0$ such that $\|g - g_*\| \leqq \delta$ implies $|\phi(f+g) - \phi(f+g_*)| \leqq \epsilon$. Choose η_1 such that $(s, t) \in \eta_1$ implies $|g_*(s) - g_*(t)| < \delta/2$. Let η satisfy $\eta \circ \eta \subset \eta_1 \cap w$. Set $F_1 = \mathrm{Cl}(\eta(A))$ and $F_2 = [(\eta_1 \cap w)(A)]^c$; hence Lemma 1 implies $F_1 \cap F_2 = \emptyset$. Choose $u \in C$ such that $u(F_1) = 0$ and $u(F_2) = 1$. Let $g = ug_*$; hence $\|g\| \leqq \|g_*\|$ since $\|u\| = 1$. If $s \in F_1$ then $u(s) = 0$; hence g has support in $R(A, \eta)^c \cap R(F, w)$. If $s \in F_2$ then $g(s) = g_*(s)$. If $s \notin F_2$ then there exists $t \in A$ such that $(s, t) \in w \cap \eta_1$; hence $|g_*(t) - g_*(s)| = |g_*(s)| \leqq \delta/2$. Therefore $|g(s) - g_*(s)| = |u(s)g_*(s) - g_*(s)| \leqq \delta$. Thus $\|g - g_*\| \leqq \delta$; hence $|\phi(f+g) - \phi(f+g_*)| \leqq \epsilon$.

LEMMA 3. *Let f and A be as in Lemma 2. Then*

$$\lim_{w \to \Delta} \sup_{g \in R(A, w, h)} |\phi(f + g) - \phi(f)| = 0.$$

PROOF. Assume false. Therefore there exists $\epsilon > 0$ such that for any η there exists $w \subset \eta$ and $g \in R(A, w, h)$ such that $|\phi(f+g) - \phi(f)| > \epsilon$.

[2] The referee has pointed out that the proof of Lemma 11 [1] (also Lemma 2: *Additive functionals on L_p spaces*, Canad. J. Math. 18 (1966), 1264–1271) only proves convergence in measure. However, the method of proof can be modified to yield convergence a.e.

Choose w_1 and $g_1^* \in R(A, w_1, h)$ such that $|\phi(f+g_1^*)-\phi(f)| > \epsilon$. By Lemma 2 there exists η_1 and g_1 such that g_1 has support in $R(A, \eta_1)^c \cap R(A, w_1)$ and $|\phi(f+g_1)-\phi(f)| > \epsilon$. Now there exists $w_2 \subset \eta_1$ and $g_2^* \in R(A, w_2, h)$ such that $|\phi(f+g_2^*)-\phi(f)| > \epsilon$. By Lemma 2 there exists η_2 and g_2 such that g_2 has support in $R(A, \eta_2)^c \cap R(A, w_2)$ and $|\phi(f+g_2)-\phi(f)| > \epsilon$. Proceeding inductively, we obtain g_i with disjoint supports such that $|\phi(f+g_i)-\phi(f)| > \epsilon$, $1 \leqq i \leqq n$. It is no loss of generality to assume $\phi(f+g_i)-\phi(f) > \epsilon$ (or $< -\epsilon$), $1 \leqq i \leqq n$, for otherwise we could choose a subsequence $g_{i_1}, \cdots, g_{i_n}$. Now (C3) implies

$$\phi\left(f + \sum_{i=1}^{n} g_i\right) - \phi(f) = \sum_{i=1}^{n} [\phi(f + g_i) - \phi(f)] > n\epsilon;$$

hence $\phi(f+\sum_{i=1}^{n} g_i) > \phi(f) + n\epsilon$ for all n. Since

$$\left\| f + \sum_{i=1}^{n} g_i \right\| \leqq \|f\| + |h|,$$

(C2) is contradicted for n sufficiently large.

REMARK. We shall now assume $\phi(0)=0$. (C3) then implies $\phi(f_1+f_2) = \phi(f_1)+\phi(f_2)$ if $f_1 f_2 \equiv 0$.

LEMMA 4. *Let* $f \in P(A, w, h)$. *Then* $\lim_{w \to \Delta} \phi(f) = \lambda_h(A)$ *exists.*

PROOF. It suffices to show that for $\epsilon > 0$ there exists w such that $w_1 \subset w$ and $w_2 \subset w$ imply $|\phi(f_1)-\phi(f_2)| < \epsilon$ where $f_i \in P(A, w_i, h)$, $i = 1, 2$. By the above remark and Lemma 3 we can choose w' such that $g \in R(A, w', h)$ implies $|\phi(g)| < \epsilon/6$. Let w satisfy $w \circ w \subset w'$. Let w_i and f_i be as above, $i = 1, 2$. Let $\eta = \eta_{f_1} \cap \eta_{f_2}$; hence $f_i(s) = h$ for $s \in \eta(A)$, $i = 1, 2$. Lemma 3 implies there exists $w_* \subset w \cap \eta$ such that $v \in R(A, w_*, h)$ implies $|\phi(f_i - v) - \phi(f_i)| < \epsilon/3$, $i = 1, 2$. Let η_1 satisfy $\eta_1 \circ \eta_1 \subset w_*$ and choose η_2, η_3, and η_4 such that $\eta_{i+1} \circ \eta_{i+1} \subset \eta_i$, $i = 1, 2, 3$. Let $F_1 = \mathrm{Cl}(\eta_4(A))$ and $F_2 = \eta_1(A)^c$. Lemma 1 implies $F_1 \cap \eta_3(A)^c = \varnothing$. Let $u_1(F_1) = 1$ and $u_1(\eta_3(A)^c) = 0$. Lemma 1 also implies $\mathrm{Cl}(\eta_2(A)) \cap F_2 = \varnothing$. Let $u_2(F_2) = 1$ and $u_2(\mathrm{Cl}(\eta_2(A))) = 0$. We note that $u_1 u_2 \equiv 0$. Since $\eta_3 \subset \eta$ and $f_i(s) = h$ for $s \in \eta(A)$ and $u_1(s) = 0$ for $s \notin \eta_3(A)$, we have $u_1 f_i = h u_1$, $i = 1, 2$. Let $y_i = (u_1 + u_2) f_i = h u_1 + g_i$ where $g_i = u_2 f_i$, $i = 1, 2$. Since $u_1 u_2 \equiv 0$ we have $\phi(y_i) = \phi(h u_1) + \phi(g_i)$, $i = 1, 2$; hence $|\phi(y_1) - \phi(y_2)| \leqq |\phi(g_1)| + |\phi(g_2)|$. We let $v_i = f_i - y_i$; hence $y_i = f_i - v_i$. We have $v_i = (1 - u_1 - u_2) f_i$; hence $v_i(s) \neq 0$ only if $s \in F_1^c \cap F_2^c$. Now

$$F_1^c \cap F_2^c = \mathrm{Cl}(\eta_1(A))^c \cap \eta_1(A) \subset A^c \cap \eta_1(A) = R(A, \eta_1) \subset R(A, w_*);$$

hence $|\phi(y_i) - \phi(f_i)| < \epsilon/3$ by choice of w_*. Since $g_i = u_2 f_i$, we have

$g_i(s)=0$ if $s\in \mathrm{Cl}(\eta_2(A))$. Also $g_i(s)=0$ if $s\notin w(A)$. Hence $g_i(s)\neq 0$ only if $s\in \mathrm{Cl}(\eta_2(A))^c\cap w(A)$. Now

$$\mathrm{Cl}(\eta_2(A))^c \cap w(A) \subset A^c \cap w(A) = R(A, w) \subset R(A, w');$$

hence $|\phi(g_i)| < \epsilon/6$ by choice of w'. Therefore we conclude

$$\begin{aligned} &|\phi(f_1) - \phi(f_2)| \\ &\qquad \leqq |\phi(f_1) - \phi(y_1)| + |\phi(y_1) - \phi(y_2)| + |\phi(y_2) - \phi(f_2)| \\ &\qquad < \epsilon/3 + |\phi(g_1)| + |\phi(g_2)| + \epsilon/3 < \epsilon. \end{aligned}$$

We have now verified the existence of $\lambda_h(A)$, $A\subset S$, for each h. The remainder of the proof follows [2] as outlined above with appropriate translation into uniformity notation.

References

1. R. V. Chacon and N. Friedman, *Additive functionals*, Arch. Rational Mech. Anal. **18** (1965), 230–240.

2. N. Friedman and M. Katz, *A representation theorem for additive functionals*, Arch. Rational Mech. Anal. **21** (1966), 49–57.

University of New Mexico and
State University of New York at Albany