ON THE NUMBER OF BINARY DIGITS IN A MULTIPLE OF THREE

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A casual examination of the multiples of three, 3, 6, 9, 12, 15, 18, 21, 24, 27, $\cdots$ written to the base two

11, 110, 1001, 1100, 1111, 10010, 10101, 11000, 11011, $\cdots$

shows a definite preponderance of those containing an even number of *one* digits over those containing an odd number. Indeed L. Moser has conjectured that this strange behavior persists forever and that, up to any point, the "evens" are more numerous than the "odds." Recent numerical studies by I. Barrodale and R. MacLeod bear out this conjecture up to 500,000 and indeed show a definite, though slightly undulatory, increase of the excess of evens over odds. [Up to 500,000 this excess numbers around 17000.]

The purpose of this note is to prove Moser's conjecture as well as to verify the trends indicated by the Barrodale-MacLeod study.

Let us define then, $D(n)$ = the number of one digits in the binary expansion of n,

$$S(N) = \sum_{0 \leq j \leq N/3} (-1)^{D(N-3j)}, \qquad \alpha = \frac{\log 3}{\log 4},$$

and $\lambda(n) = n^{-\alpha} S(3n)$.

We will prove the following

THEOREM. *For all n, $1/20 < \lambda(n) < 5$, but $\lim_{n\to\infty} \lambda(n)$ does not exist.*

This verifies the numerical study perfectly by displaying a growth of the order of n^α but undulating between two constant multiples of n^α. Of course just the fact that these constants are positive proves Moser's conjecture.

To prove our theorem we will obtain a semiexplicit expression for $S(N)$ from which the required results follow easily. Observe that if $n < 2^k$, then $D(n+2^k) = 1 + D(n)$. Thus if we denote $\rho = 3[n/3] + 3 - n$, we have

Received by the editors August 1, 1968.

[1] Partially supported by the National Research Council of Canada during a visit to the University of Alberta.

$$\sum_{0\leq j\leq (n+2^k)/3} (-1)^{D(n+2^k-3j)}$$
$$= \sum_{0\leq j\leq n/3} (-1)^{D(n+2^k-3j)} + \sum_{0\leq \nu\leq (2^k-\rho)/3} (-1)^{D(2^k-\rho-3\nu)}$$
$$= - \sum_{0\leq j\leq n/3} (-1)^{D(n-3j)} + \sum_{0\leq \nu\leq (2^k-\rho)/3} (-1)^{D(2^k-\rho-3\nu)},$$

which is to say

$$S(n+2^k) = S(2^k-\rho) - S(n). \tag{1}$$

We can now iterate the relation (1). Indeed if we write $N=2^{k_1}+2^{k_2}+\cdots+2^{k_j}$, $k_1>k_2\cdots>k_j\geqq 0$, and note that $S(2^k)=S(2^k-3)-1$, we obtain

$$\begin{aligned} S(N) &= S(2^{k_1}-\rho_1) - S(2^{k_2}-\rho_2) + \cdots \\ &\quad + (-1)^{j-1}S(2^{k_j}-\rho_j) + (-1)^j \end{aligned} \tag{2}$$

where the ρ_i are chosen as 1, 2, or 3 so as to make $\rho_i+2^{k_{i+1}}+2^{k_{i+2}}+\cdots+2^{k_j}$ divisible by 3.

Next we need a formula for the $S(2^k-\rho)$ which appear in (2). This is given as follows:

$$S(2^{2m}-1) = 2\cdot 3^{m-1}, \tag{3}$$
$$S(2^{2m}-2) = -3^{m-1}, \tag{4}$$
$$S(2^{2m}-3) = -3^{m-1}, \tag{5}$$
$$S(2^{2m+1}-1) = -3^m, \tag{6}$$
$$S(2^{2m+1}-2) = 3^m, \tag{7}$$
$$S(2^{2m+1}-3) = 0. \tag{8}$$

These may be proved analytically via the generating function $\prod(1-x^{2^\nu})=\sum(-1)^{D(n)}x^n$. More specifically we use $\prod_{\nu=0}^{k-1}(1-x^{2^\nu})=\sum_{n=0}^{2^k-1}(-1)^{D(n)}x^n$ and obtain, for $\rho>0$, that

$$S(2^k-\rho) = \frac{1}{2\pi i}\int_{|x|=1/2} \frac{\prod_{\nu=0}^{k-1}(1-x^{2^\nu})}{1-x^3}\,\frac{dx}{x^{2^k-\rho+1}}.$$

If $\rho\leqq 3$, furthermore, the integrand is $O(1/x^2)$ at ∞. Thus if we move the contour to a large circle $|x|=R$, record the residues at the cube roots of unity ω and ω^2, and let $R\to\infty$ we obtain the above results. [It seems perhaps that an elementary proof could also be given.][2]

[2] ADDED IN PROOF. Recently S. Klein has supplied such a proof. It is an induction on (3)–(8) using (1).

We now apply our formula (2) to the case $N=3n$. What is special here is the fact that $3\mid 2^{k_1}-\rho_1$ and $3\nmid 2^{k_2}-\rho_2$. An examination of (3) through (8), therefore, tells us that

$$S(2^{k_1}-\rho_1) \geqq 3^{(k_1-1)/2}, \qquad S(2^{k_2}-\rho_2) \leqq 0. \tag{9}$$

Again (3) through (8) imply in general that

$$\left| S(2^k-\rho)\right| \leqq (2/3)3^{k/2}. \tag{10}$$

Combined use of (9) and (10) in (2) gives

$$\begin{aligned} S(3n) &\geqq 3^{(k_1-1)/2} - \frac{2}{3}3^{k_3/2} - \frac{2}{3}3^{k_4/2} - \cdots \\ &\geqq 3^{(k_1-1)/2} - \frac{2}{3}3^{(k_1-2)/2} - \frac{2}{3}3^{(k_1-3)/2} - \cdots \\ &\geqq 3^{(k_1-1)/2}\left(1 - \frac{2}{3\sqrt{3}} - \frac{2}{3\sqrt{3}^2} - \frac{2}{3\sqrt{3}^3} - \cdots\right) \\ &= 3^{(k_1-1)/2}\left[1 - \frac{2}{3(\sqrt{3}-1)}\right] \geqq \frac{1}{20}3^{k_1/2} \geqq \frac{1}{20}n^\alpha \end{aligned} \tag{11}$$

and this gives the lower bound on $\lambda(n)$.

To obtain the upper bound simply insert (10) into (2) to derive

$$\begin{aligned} S(3n) &\leqq \frac{2}{3}\left[3^{k_1/2} + 3^{k_2/2} + \cdots\right] \leqq \frac{2}{3}\left[3^{k_1/2} + 3^{(k_1-1)/2} + \cdots\right] \\ &< \frac{2}{3-\sqrt{3}}3^{k_1/2} \leqq \frac{2}{3-\sqrt{3}}(3n)^\alpha \leqq 5n^\alpha \end{aligned} \tag{12}$$

as required.

Finally note from (3) that through the subsequence $n=(2^m-1)/3$, $\lambda(n)\to 2/3$ and that by (7) the subsequence $n=(2^{2m+1}-2)/3$ gives $\lambda(n)\to 1/\sqrt{3}$. This completes the proof.

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