ON STRONG MEASURABILITY OF BANACH VALUED FUNCTIONS[1]

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The purpose of this note is to point out that, as an immediate consequence of our Theorem 11 in [3], we get an interesting measurability property for some Banach valued functions.

As a corollary, we get an improvement of Theorem 4 of [3], to the extent that the separability condition in [3] can be dropped.

1. **Statement of the Theorem.** Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space, μ being a finite positive measure on the σ-field $\mathfrak{F}$ of subsets of Ω. Let $\boldsymbol{B}$ be a Banach space, and $\mathfrak{S}$ a set of convex equilibrated weakly compact subsets of $\boldsymbol{B}$. A mapping f from Ω into $\boldsymbol{B}$ is said to be strongly μ-measurable if it takes its values in a separable subset of $\boldsymbol{B}$ and, for every ball B in $\boldsymbol{B}$, $f^{-1}(B)$ belongs to the μ-completion of $\mathfrak{F}$. We denote by $\boldsymbol{B}'$ the Banach dual space of $\boldsymbol{B}$.

Then we have the following

THEOREM. *Let f be a mapping of $(\Omega, \mathfrak{F}, \mu)$ into the Banach space $\boldsymbol{B}$, with the following properties:*

(i) *For every strictly positive number ϵ, there exists $\Omega_\epsilon \in \mathfrak{F}$ and $Q_\epsilon \in \mathfrak{S}$ such that $\mu(\Omega \backslash \Omega_\epsilon) \leqq \epsilon$, $f(\Omega_\epsilon) \subset Q_\epsilon$.*

(ii) *For every continuous linear form x' on $\boldsymbol{B}$, the mapping $\omega \hookrightarrow x' \circ f(\omega)$ (denoted $\langle f, x' \rangle$) from $(\Omega, \mathfrak{F}, \mu)$ into $\boldsymbol{R}$ (set of real numbers) is μ-measurable.*

Then, there exists a strongly μ-measurable mapping $\bar{f}$ from $(\Omega, \mathfrak{F}, \mu)$ into $\boldsymbol{B}$, such that for every $x' \in \boldsymbol{B}'$, $\langle f, x' \rangle = \langle \bar{f}, x' \rangle$ μ a.e.

2. **Proof of the Theorem.** For every $F \in \mathfrak{F}$, with $F \subset \Omega_\epsilon$, it is well known (cf. [1] for example) that the weak integral $\int_F f d\mu$ exists and

$$m(F) = \int_F f d\mu \in \mu(F) Q_\epsilon.$$

Let us then apply Theorem 11 of [3] to the restriction of μ and $\boldsymbol{m}$ to Ω_ϵ. We see then that there exists a strongly measurable $\bar{f}_\epsilon$ such that $\boldsymbol{m}(F) = \int_F \bar{f}_\epsilon d\mu$ for every $F \subset \Omega_\epsilon$. Hence, for every $x' \subset \boldsymbol{B}'$ we have $\langle \bar{f}_\epsilon, x' \rangle = \langle f, x' \rangle$ a.e. on Ω_ϵ. The theorem follows by considering a sequence Ω_n of disjoint subsets of Ω, such that $\mu(\bigcup_n \Omega_n) = \mu(\Omega)$ and defining $\bar{f}$ by its restriction $\bar{f}_n$ to each Ω_n.

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3. **Application to Banach valued martingales.** Let us go back to the terminology of [**3**], considering the strong martingale $(\mathfrak{F}_\alpha, f_\alpha)_{\alpha\in I}$ as it is described by the assumptions of Theorem 4, but dropping the separability assumption on the Banach space V. The function f_∞ which occurs at the beginning of the proof in [**3**] verifies the assumptions of the above theorem, and can be then replaced by a $\bar{f}_\infty$ which is strongly measurable and possesses the same properties with respect to the martingale $(\mathfrak{F}_\alpha, f_\alpha)_{\alpha\in I}$.

In the special case when V is reflexive, we can thus state

COROLLARY. *Let $(\mathfrak{F}_\alpha, f_\alpha)_{\alpha\in I}$ be a strong martingale with values in a reflexive Banach space V, I being any directed ordered set. In order for $(f_\alpha)_{\alpha\in I}$ to converge in $L^1(\Omega, \mathfrak{F}, \mu, V)$ to a strongly integrable function f_∞, it is necessary and sufficient that the family (f_α) be terminally uniformly integrable.*

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