FREDHOLM OPERATORS: A COUNTEREXAMPLE

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In 1960 Pietsch [1] gave an example of a locally convex linear topological space for which the set of Fredholm operators is not open. It is the purpose of this note to show that the Fredholm class is not open for any weak locally convex space. A Hausdorff locally convex space is called a weak locally convex space if every null-neighborhood contains a closed subspace of finite codimension.

A continuous linear operator T mapping a linear topological space X into itself is called a Fredholm operator if it enjoys the following properties:

(1) The null space $N(T)$ is finite dimensional,

(2) the range space $R(T)$ is finite codimensional,

(3) T is relatively open, and

(4) $R(T)$ is closed.

The class of Fredholm operators on X is denoted by $\Sigma(X)$.

Let X be a Hausdorff locally convex space and $L(X)$ the space of continuous linear operators on X. $L(X)$ is topologized with the topology of uniform convergence on bounded subsets of X.

THEOREM. *Let X be a weak locally convex infinite-dimensional space, then $\Sigma(X)$ is not open in $L(X)$.*

PROOF. By way of contradiction, suppose $\Sigma(X)$ is open $L(X)$. Accordingly, there exists a bounded set B in X and a balanced null-neighborhood U such that $I+N(B, U)\subset\Sigma(X)$ where I is the identity operator and $N(B, U)=\{T\in L(X): T(B)\subset U\}$. Since X is a weak locally convex space, there exists a closed subspace Y of finite codimension such that $Y\subset U$. Let P be a continuous projection of X onto Y [2, p. 22]. Then $P\in N(B, U)$ which implies that $-P\in N(B, U)$. But then

$$I - P \in I + N(B, U)$$

which is impossible since $R(I-P)$ is finite dimensional.

Since the space of all complex sequences with the usual topology is a weak locally convex space, Pietsch's example [1, p. 355] is an immediate consequence of the above theorem. The following corollary is easily established.

Received by the editors May 20, 1968.

COROLLARY. *Let X be an infinite-dimensional Banach space with its weak topology, then $\Sigma(X)$ is not open in $L(X)$.*

REFERENCES

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2. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.

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