AMPLE VECTOR BUNDLES ON ALGEBRAIC SURFACES

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The positivity of the Chern classes c_i of an ample vector bundle on an algebraic surface is studied. Notably the inequality $0 < c_2 < c_1^2$ is established. This inequality was conjectured by Hartshorne [5] and Griffiths [1] (for compact, complex manifolds).

Let X be a scheme of finite type over an algebraically closed field, E a vector bundle on X (i.e., a locally free sheaf of constant, finite rank), and $S^n(E)$ the nth symmetric power of E.

DEFINITION (HARTSHORNE [5]) 1. The bundle E is *ample* if for every coherent sheaf F on X, there is an integer $N > 0$, such that for every $n \geqq N$, the sheaf $F \otimes S^n(E)$ is generated by its global sections.

PROPOSITION (HARTSHORNE [5]) 2. *Consider the following conditions*:

(i) *The bundle E is ample.*

(ii) *Let $P = \boldsymbol{P}(E)$ be the associated projective bundle and $L = O_P(1)$ the tautological line bundle. Then L is ample on P.*

(iii) *For every coherent sheaf F on X, there exists an integer $N > 0$, such that for $n \geqq N$ and $q \geqq 1$*

$$H^q(X, F \otimes S^n(E)) = 0.$$

Then (i) *and* (ii) *are equivalent and they are implied by* (iii). *If further, X is complete, then* (i), (ii) *and* (iii) *are all equivalent.*

THEOREM 3. *Let X be an irreducible, nonsingular surface which is projective over an algebraically closed field, and let $A(X)$ be the Chow $\boldsymbol{R}$-algebra of cycles modulo numerical equivalence. Let E be a vector bundle of rank $r \geqq 2$ on X, and let $c_1, c_2 \in A(X)$ be the Chern classes of E. Assume E is ample. Then, $c_2 > 0$ and $c_1^2 - c_2 > 0$.*

PROOF. Since E is ample on X, then $O_P(1)$ is ample on $P = \boldsymbol{P}(E)$. Hence, by [EGA II, 4.4.1, 4.4.2 and 4.4.10], there exist an integer $n \geqq 2$ and a projective embedding, $j: P \rightarrow Y = \boldsymbol{P}_k^N$ such that $O_P(n) = j^* O_Y(1)$. For this embedding, let S be the Chow variety parametrizing the 2-dimensional sections of P by linear spaces and T the subvariety of S corresponding to those sections which meet a given fiber of $P \rightarrow X$ in infinitely many points. As $n \geqq 2$, the codimension of T in S is at least 3.

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Let H be a general 2-dimensional linear section of P. By the principal of counting constants, the map $H \to X$ has finite fibers; so, it is finite by [EGA III, 4.4.2]. Further, H is irreducible and nonsingular by Bertini's theorems [EGA V].

Let l be the class of $O_P(1)$ in the Chow algebra $A(P)$. By [2] or [3.1], $A(P)$ is generated over $A(X)$ by l modulo the relation,

$$l^r - c_1 l^{r-1} + c_2 l^{r-2} = 0. \tag{3.1}$$

Let $a \in A^1(X)$. Then, $(l - c_1) \cdot l^{r-1} \cdot a = -c_2 \cdot a \cdot l^{r-2} = 0$. Let $h \in A(P)$ be the class of H; then, $h = (nl)^{r-1}$. Therefore,

$$(l - c_1) \cdot a \cdot h = 0. \tag{3.2}$$

Let $i: H \to P$ be the inclusion map and i_*, i^* the maps induced on the Chow algebras. Then, $i_* i^* b = b \cdot h$ for $b \in A(P)$. In view of (3.2), it follows that for any $a \in A^1(X)$,

$$i^*((l - c_1) \cdot a) = 0. \tag{3.3}$$

The Lefschetz hyperplane theorem [**3.2**, XIII, 4.6 (iii)$\Leftrightarrow$(vi)] implies that $i^*(l - c_1) \neq 0$ because $(l - c_1) \neq 0$. Let $a \in A^1(X)$ be the class of an ample line bundle. Since H is finite over X, then $a \cdot 1_H \in A^1(H)$ is the class of an ample line bundle by [EGA II, 5.1.12]. In view of (3.3), the Hodge index theorem [**3.2**, XIII, 7.1] asserts that $0 > i^*(l - c_1)^2$; thence, by (3.1) and (3.3) with $a = c_1$, it follows that $0 > -c_2 \cdot 1_H$.

Similarly,[2] since $i^* l$ is the class of an ample line bundle on H, then $0 < i^* l^2$; thence, by (3.1) and (3.2) with $a = c_1$, it follows that $0 < (c_1^2 - c_2) \cdot 1_H$.

Remark 4. With the more general theory of Chern classes developed in [**3.1**], the same reasoning establishes that $c_2 > 0$ and $c_1^2 - c_2 > 0$ for an ample bundle E on an arbitrary surface X. Consequently, on a projective algebraic scheme Y of arbitrary dimension, an ample bundle E has classes c_2 and $c_1^2 - c_2$ which have positive intersection number with every surface X on Y.

Example 5. Under the conditions of Theorem 3, the inequality $c_1^2 - c_2 > 0$ is best possible in the following sense. There exists a sequence of ample, rank 2 bundles E_n on X, such that for all $\epsilon > 0$, $(c_1^2 - (1+\epsilon) c_2(E_n))$ equals $-\epsilon n^2 d + \cdots$ with $d = \deg(X)$, so it tends to $-\infty$ as $n \to \infty$.

To construct E_n, fix a surjection $\alpha_n: O_X^{\oplus 3} \to O_X(n)$. Let F_n be the

[2] This line is due to Hartshorne who commented in private on the proof that $c_2 > 0$.

dual of the kernel of α_n and $E_n = F_n(1)$. Then E_n is a rank 2 bundle and there is an exact sequence

$$0 \to O_X(1-n) \to O_X(1)^{\oplus 3} \to E_n \to 0.$$

Hence, $c_1^2(E_n) = (n+2)^2 d$ and $c_2(E_n) = (n^2+n+1)d$. Finally, since E_n is a quotient of a direct sum of ample line bundles, E_n is ample [**5**, (2.2)].

In characteristic $p > 0$, there are two new notions extending the notion of ampleness for line bundles: Let $f: X \to X$ be the Frobenius (pth-power) endomorphism and f_n the nth iterate of f.

DEFINITION (HARTSHORNE [**5**]) 6. (i) The bundle E is *p-ample* if for every coherent sheaf F, there is an integer $N > 0$, such that for every $n \geqq N$, the sheaf $F \otimes f_n^* E$ is generated by its global sections.

(ii) The bundle E is *cohomologically p-ample* if for every coherent sheaf F on X, there is an integer $N > 0$, such that for $n \geqq N$ and $q \geqq 1$, $H^q(X, F \otimes f_n^* E) = 0$.

REMARK 7. (i) Assume X is quasi-projective. Then any coherent sheaf F is a quotient of a sheaf of the form $O_X(-m)^{\oplus M}$ for $m, M \gg 0$. It follows that for the Definitions 6 (as well as for the analogous formulations of ampleness) it suffices to verify the condition on sheaves of the form $F = O_X(-m)$ for $m \gg 0$.

(ii) Hartshorne [**5**, (6.3)] proves that p-ample bundles are ample. He conjectures the converse, and proves it for line bundles and for curves [**5**, (7.3)].

EXAMPLE 8. A p-ample bundle on a complete scheme need not be cohomologically p-ample. In fact, the rank 2 bundles E_n constructed in (5) are p-ample being quotients of direct sums of p-ample bundles [**5**, (6.4)]; however, for $n \geqq 2$, (although quotients of cohomologically p-ample bundles) they are not cohomologically p-ample because for $m \gg 0$, $H^1(X, f_m^* E_n)$ equals $H^2(X, O_X(-m(n-1)))$, which is >0 by [**6**, p. 944].

PROPOSITION 9. *Suppose X is quasi-projective and E is cohomologically p-ample. Then E is p-ample.*

PROOF. In view of (7)(i), fix an integer $m > 0$ and let $G_n = (f_n^* E)(-m)$.

Let $x \in X$ be a closed point. Then there is an N such that the stalk $(G_N)_x$ is generated by global sections. Indeed, it suffices to show that the map $H^0(X, G_N) \to H^0(X, G_N \otimes k(x))$ is surjective. However, by hypothesis, there is an N such that $H^1(X, I_x \otimes G_N) = 0$ where I_x is the ideal defining $\{x\}$. There is, therefore, a neighborhood U of x in which G_N is generated by global sections.

Let $n=N+t$, with $t\geqq 0$. Then, $(f_n^*E)(-mp^t)=f_t^*G_N$ is generated in U by global sections. However, for any sheaf G and $r\geqq 0$, G is a quotient of $G(-r)^{\oplus s}$ for suitable s. Thus, G_n is generated in U by global sections. By quasi-compactness, it follows that E is p-ample.

LEMMA 10. *Suppose X is integral, quasi-projective and of dimension r and E is p-ample. Then for some $a>0$,*

$$h^0(X, f_n^*E) \geqq ap^{rn} + \cdots .$$

PROOF. Take N such that $(f_N^*E)(-1)$ is generated by global sections. It follows that there is a map $\beta\colon O_X(1)\to f_N^*E$ which is a split-injection on an open set. Let $n=N+t$, with $t\geqq 0$. Then, O_X being torsion free, $f_t^*\beta\colon O_X(p^t)\to f_n^*E$ is an injection. Thus, $h^0(X, f_n^*E)\geqq h^0(X, O_X(p^t))$; whence the conclusion.

THEOREM (HIRONAKA[3]) 11. *Let X be an integral (nonsingular) surface which is projective over an algebraically closed field of characteristic $p>0$, E a cohomologically p-ample bundle on X, and c_1, c_2 the Chern classes of E modulo numerical equivalence. Then, $c_1^2-2c_2>0$.*

PROOF. For any bundle E on X, the Riemann-Roch theorem implies that $\chi(f_n^*E)=((c_1^2-2c_2)/2!)p^{2n}+\cdots$. Suppose E is cohomologically p-ample. Then, in view of (9) and (10), E is p-ample and $c_1^2-2c_2>0$.

REFERENCES

1. P. Griffiths, *Hermitian differential geometry and positive vector bundles*, (to appear).

2. A. Grothendieck, *Sur quelques propriétés fondamentales en théorie des intersections*, Séminaire C. Chevalley, École Norm. Sup., Paris, 1958.

3. A. Grothendieck et al., *Séminaire de géométrie algébrique*, 1966–1967, North-Holland, Amsterdam, (to appear).

3.1. Exposes of Berthelot.

3.2. Expose of Kleiman.

4 [EGA]. A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math., No. 4, (1960).

5. R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math., No. 29, (1966), 63–94.

6. S. Kleiman, *On the vanishing of $H^n(X, F)$ for an n-dimensional variety*, Proc. Amer. Math. Soc. 18 (1967), 940–944.

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[3] This result was in essence contained in a private communication from Hironaka.