AN EXISTENCE THEOREM FOR THE REDUCED WAVE EQUATION

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We consider here the two dimensional problem of scattering of a wave by a finite set of smooth, finite, nonintersecting arcs. Let the points in E^2 be denoted by $z=x+iy$. Let L_i, $i=1, \cdots, n$ be arcs given by

$$z = x_i(t) + iy_i(t), \qquad 0 \leqq t \leqq 1, \qquad i = 1, \cdots, n.$$

We denote the point $x_i(0)+iy_i(0)$ by a_i and the point $x_i(1)+iy_i(1)$ by b_i. We assume that the functions $x_i(t)$ and $y_i(t)$ have Hölder continuous second derivatives and that the arcs L_i do not intersect. We denote the union of the L_i's by L and the open set E^2-L by G. We seek a function $u_s(x, y)$ which satisfies the following conditions:

(a) u_s is continuous in E^2,

(b) in G, u_s satisfies the reduced wave equation

$$\frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} + k^2 u_s = 0, \qquad k \neq 0, \quad \operatorname{Re} k \geqq 0, \quad \operatorname{Im} k \geqq 0, \tag{1}$$

(c) u_s satisfies the radiation condition

$$\lim_{\rho\to\infty} \int_{|z|=\rho} \left| \frac{\partial u_s}{\partial |z|} - iku_s \right|^2 ds = 0, \tag{2}$$

(d) on L, $u_s+u_0=0$

where u_0 is a prescribed function (the incident wave) which satisfies (1) in all of E^2.

The purpose of this note is to prove the following

THEOREM. *There exists a* (*unique*) *function* u_s *satisfying conditions* (a)–(d) *above.*

Uniqueness follows from the work of Levine [2]. (Levine proves his uniqueness theorem in the three dimensional case, however his proof can easily be modified so as to apply here.) Here we will prove existence. Our method is similar to that of Leis [1] who considered the case of scattering by a piecewise smooth closed contour.

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PROOF OF THE EXISTENCE THEOREM. Let $z_0=x_0+iy_0$ and $r=|z-z_0|$. Define

$$Q(z, z_0) = (i/4)H_0^{(1)}(kr), \tag{3}$$

where $H_0^{(1)}$ denotes the zero order Hankel function of the first kind. Then for fixed z_0 and $z\neq z_0$ Q satisfies (1) and (2). For small r we have

$$(i/4)H_0^{(1)}(kr) = (1/2\pi)\log(1/r) + h(r), \tag{4}$$

where $h(r)$ and $h'(r)$ are finite at $r=0$ (i.e. $Q(z, z_0)$ is the fundamental outgoing solution of the reduced wave equation).

We seek a solution to the problem of the form

$$u_s(z) = \int_L \phi(z_0)Q(z, z_0)ds_0. \tag{5}$$

We assume that the unknown function ϕ defined on L is Hölder continuous in the neighborhood of every interior point of the arcs L_i and if c_i is one of the endpoints of L_i ($c_i=a_i$ or b_i) we have

$$|\phi(z_0)| \leqq K/|z_0 - c_i|^\alpha, \quad z_0 \in L_i \quad 0 \leqq \alpha < 1.$$

Functions satisfying the above conditions will be said to belong to class h. (We are motivated to seek ϕ out of this class of functions by examining the solution of the problem of diffraction by a half line.) Then the function u_s defined by (5) will satisfy conditions (a), (b), and (c) of the problem. To satisfy the boundary condition (d) we must have

$$-u_0(z) = \int_L \phi(z_0)Q(z, z_0)ds_0, \qquad z \in L. \tag{6}$$

By (4) $Q(z, z_0)$ has a logarithmic singularity at $z=z_0$. However, by a theorem in [**3**, p. 31] we may differentiate the right-hand side of (6) by differentiating formally under the integral sign and interpreting the result as a Cauchy Principal Value. Now

$$\frac{\partial Q}{\partial s} = -\frac{1}{2\pi}\frac{d\log r}{ds} + h'(r)\frac{dr}{ds}\cdot$$

$r=(z-z_0)e^{-i\nu}$ where ν is the argument of the difference $z-z_0$. Hence

$$\frac{d\log r}{ds} = \frac{1}{z-z_0}\frac{dz}{ds} - i\frac{d\nu}{ds} = \frac{1}{z-z_0}e^{i\theta(z)} - i\frac{d\nu}{ds}$$

where $\theta(z)$ is the angle between the x axis and the tangent to L at z.

Further $ds_0 = \exp(-i\theta(z_0))dz_0$. Thus the result can be written in the form

$$f(z) \equiv -\frac{\partial u_0}{\partial s}(z) = \int_L \phi(z_0)\frac{K(z, z_0)}{z_0 - z}\,dz_0, \tag{7}$$

with

$$K(z, z_0) = \left\{\frac{1}{2\pi}e^{i\theta(z)} + \frac{i}{2\pi}\frac{dv}{ds}(z - z_0) + h'(r)\frac{dr}{ds}(z_0 - z)\right\}e^{-i\theta(z_0)}.$$

$K(z, z_0)$ is Hölder continuous with respect to both variables on each of the closed arcs L_i and $K(z_0, z_0) \neq 0$, $z_0 \in L$. It is required then to find a solution ϕ in the class h of the singular integral equation (7). ϕ is to be a solution in the sense that (7) holds for z on L except perhaps at end points.

Singular integral equations of the type (7) are treated in Chapter 14 of [**3**].[2] If we write (7) as

$$K\phi = f, \tag{8}$$

we may state the results of [**3**] as applied to the present situation as follows.

1. The necessary and sufficient conditions of solubility in class h of the equation (8) are $\int_L f(z)\Psi_j(z)dz = 0$ where Ψ_j $(j = 1, \cdots, k')$ is a complete system of linearly independent solutions in a certain class h' of an adjoint equation $K'\psi = 0$. (The definitions of K' and h' need not concern us here.)

2. If k is the number of linearly independent solutions of the class h of the homogeneous equation $K\phi = 0$ and k' is the number of linearly independent solutions of the class h' of the adjoint equation $K'\psi = 0$, then $k - k' = n$.

In the present case we shall show that $k = n$. Hence $k' = 0$ and thus (8) will have a solution for every (Hölder continuous) f.

Let ϕ_j be a solution of class h of $K\phi = 0$. Then we know that the function

$$v_j(z) = \int_L \phi_j(z_0)Q(z, z_0)ds_0$$

is constant on each L_i since its derivative with respect to s vanishes on each L_i. Define c_{ij} by $v_j(z) = c_{ij}$, $z \in L_i$.

[2] In [**3**] the author considers a more general equation namely $A(z)\phi(z) + (1/\pi i) \cdot \int_L (K(z, z_0)\phi(z_0)/(z_0 - z))dz_0 = f(z)$. The basic assumption under which this equation is studied is that if $B(z) = K(z, z)$ then $A^2(z) - B^2(z) \neq 0$ for z on L. In the present case we have $A(z) = 0$. The above condition then obtains since we have $K(z, z) = 0$.

Now let $\phi_1, \cdots, \phi_k$ be a complete set of linearly independent solutions of $K\phi = 0$. We have by statement 2 above $k \geqq n$. We now prove

LEMMA 1. $k = n$ *(hence* $k' = 0$*). Furthermore the rank of the matrix* (c_{ij}) *is* n.

Suppose the lemma were false. Then we could find constants a_j, $j = 1, \cdots, k$ not all zero such that

$$\sum_{i=1}^{k} c_{ij}a_j = 0, \qquad i = 1, \cdots, n.$$

Consider the functions $\phi_0 = \sum_{j=1}^{k} a_j\phi_j$ and $v_0 = \int_L \phi_0(z_0)Q(z, z_0)ds_0$. On L_i we have

$$v_0(z) = \sum_{j=1}^{k} a_j \int_L \phi_j(z_0)Q(z, z_0)ds_0 = \sum_{j=1}^{k} a_jc_{ij} = 0.$$

Hence v_0 satisfies conditions (a), (b), (c) and vanishes on L. Thus by uniqueness $v_0 \equiv 0$. But then $\phi(z_0) = [\partial v/\partial n](z_0) = 0$ where $[\partial v_0/\partial n](z_0)$ denotes the jump in the normal derivative of v_0 across L at the point z_0. Thus $\sum_{j=1}^{k} a_j\phi_j \equiv 0$, contradicting the linear independence of the ϕ_j's. Hence we can solve (7) for any f. Let η_0 be a solution of (7) in the class h. Then we have

$$u(z) = \int_L \eta_0(z_0)Q(z, z_0)ds_0 = -u_0(z) + C_i, \qquad z \in L_i,$$

where C_i are definite constants. But by Lemma 1 we can determine constants $a_1, \cdots, a_n$ so that

$$\sum_{j=1}^{n} c_{ij}a_j = -C_i, \qquad i = 1, \cdots, n.$$

Then if $\eta = \eta_0 + \sum_{j=1}^{n} a_j\phi_j$, $u_s(z) = \int_L \eta(z_0)Q(z, z_0)ds_0$ satisfies (6) and hence satisfies conditions (a)–(d) of the problem.

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