ON THE MULTIPLIERS OF H^p SPACES

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A complex sequence $\{\lambda_n\}$ is called a multiplier of H^p into H^q if $\sum\lambda_n a_n z^n$ is in H^q whenever $\sum a_n z^n$ is in H^p. In the context of fractional integration, Hardy and Littlewood [4, p. 415] showed that if $0<p<q<\infty$, $\alpha=1/p-1/q$, and

$$\lambda_n = \frac{n!}{\Gamma(n+1+\alpha)} = n^{-\alpha} + O(n^{-\alpha-1}), \tag{1}$$

then $\{\lambda_n\}$ multiplies H^p into H^q. The question arises whether the condition $\lambda_n = O(n^{-\alpha})$ alone implies $\{\lambda_n\}$ is such a multiplier. We show here that this is true if $0<p\leqq 2\leqq q<\infty$, or if $0<p\leqq 1$ and $q=\infty$, but false otherwise. The precise results are as follows.

THEOREM 1. *If $0<p\leqq 2\leqq q<\infty$, $\alpha=1/p-1/q$, and $\lambda_n=O(n^{-\alpha})$, then $\{\lambda_n\}$ is a multiplier of H^p into H^q. The same is true if $0<p\leqq 1$ and $q=\infty$, but not if $1<p<q=\infty$. The number α is best possible: for each $a<\alpha$, there is a sequence $\{\lambda_n\}$ with $\lambda_n=O(n^{-a})$ which is not a multiplier of H^p into H^q.*

THEOREM 2. *If $0<p<q<2$, the condition $\lambda_n=O(n^{-\alpha})$ does not imply that $\{\lambda_n\}$ is a multiplier of H^p into H^q. In fact, for each number $\beta<1/p-1/2$, there is a sequence $\{\lambda_n\}$ with $\lambda_n=O(n^{-\beta})$ which is not a multiplier of H^p into H^q for any $q>0$.*

THEOREM 3. *If $2<p<q\leqq\infty$, the condition $\lambda_n=O(n^{-\alpha})$ does not imply that $\{\lambda_n\}$ is a multiplier of H^p into H^q. In fact, for each number $\beta<1/2-1/q$, there is a sequence $\{\lambda_n\}$ with $\lambda_n=O(n^{-\beta})$ which is not a multiplier of H^p into H^q for any $p<\infty$.*

PROOF OF THEOREM 1. Suppose first that $0<p\leqq 1$ and $2\leqq q\leqq\infty$. If $\sum a_n z^n$ is in H^p, then $a_n=o(n^{1/p-1})$, by a theorem of Hardy and Littlewood [4]. This and the hypothesis on $\{\lambda_n\}$ give

$$|\lambda_n a_n|^{q'} \leqq Cn^{p-2}|a_n|^p,$$

where $q'=q/(q-1)$ is the conjugate index. But by another theorem of Hardy and Littlewood [3], $\sum n^{p-2}|a_n|^p<\infty$ if $\sum a_n z^n$ is in H^p $(0<p\leqq 2)$. Thus $\{\lambda_n a_n\}\in l^{q'}$, and it follows from the Hausdorff-Young theorem that $\sum\lambda_n a_n z^n$ is in H^q.

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Next suppose $1<p\leqq 2=q$ and that $\lambda_n=O(n^{1/2-1/p})$. If $\sum a_n z^n$ is in H^p, then by Hölder's inequality, the Hardy-Littlewood theorem, and the Hausdorff-Young theorem,

$$\sum_{n=1}^{\infty} |\lambda_n a_n|^2 \leqq C \sum_{n=1}^{\infty} n^{1-2/p} |a_n|^2$$
$$\leqq C\left\{\sum_{n=1}^{\infty} n^{p-2}|a_n|^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} |a_n|^{p'}\right\}^{1/p'} < \infty.$$

Thus $\{\lambda_n\}$ multiplies H^p into H^2.

Finally, suppose $1<p\leqq 2<q<\infty$. Given $\lambda_n=O(n^{-\alpha})$, let

$$\mu_n = n^{1/2-1/q}\lambda_n.$$

Then by what we have just shown, $\{\mu_n\}$ is a multiplier of H^p into H^2. Thus to complete the proof it will suffice to show that $\{n^{1/q-1/2}\}$ multiplies H^2 into H^q. But by (1),

$$n^{1/q-1/2} = \frac{n!}{\Gamma(n+3/2-1/q)} + \nu_n, \tag{2}$$

where $\nu_n=O(n^{1/q-3/2})$. If $\sum a_n z^n$ is in H^2, then trivially $\{\nu_n a_n\}\in l^{q'}$, since $\{a_n\}$ is bounded. Hence $\{\nu_n\}$ is a multiplier of H^2 into H^q; and in view of (2) and the Hardy-Littlewood result mentioned in the opening paragraph, $\{n^{1/q-1/2}\}$ is also such a multiplier. This proves that $\{\lambda_n\}$ is a multiplier if $\lambda_n=O(n^{-\alpha})$.

That the corresponding statement is false if $1<p<q=\infty$ can be seen by considering the function

$$f(z) = (1-z)^{-1/p'} = \sum_{n=0}^{\infty} \lambda_n z^n.$$

Here $\lambda_n=O(n^{-1/p})$, but if $\{\lambda_n\}$ were a multiplier of H^p into H^∞, it would follow from a result of Caveny [1] that $f\in H^{p'}$, which is not the case.

Finally, suppose $0<a<b<\alpha$. Then $(1-z)^{-1/q-b}\in H^p$, but multiplication by a suitable sequence $\{\lambda_n\}$ with $\lambda_n=O(n^{-a})$ produces $(1-z)^{-1/q-b+a}\notin H^q$, $q\leqq\infty$. Hence α is best possible.

Corollary. *If $0<p<q<\infty$, then $\{n^{-\alpha}\}$ is a multiplier of H^p into H^q.*

Proof. Hardy and Littlewood showed that the sequence $\{\lambda_n\}$ given by (1) is a multiplier of H^p into H^q. But if $\mu_n=O(n^{-\alpha-1})$, Theorem 1 shows that $\{\mu_n\}$ multiplies H^p into H^q, provided $q\geqq 2/3$. Indeed, the condition guarantees that $\{\mu_n\}$ multiplies H^2 into H^q if $2 \leqq p < q$; while if $0 < p < q < 2$ and $q \geqq 2/3$, $\{\mu_n\}$ will multiply H^p into H^2. A similar discussion shows that $\{n^{-\alpha-k}\}$ multiplies H^p into H^q if $q \geqq 2/(2k+3)$, $k = 1, 2, \cdots$. Now, using the more precise formula

$$\lambda_n = n^{-\alpha} + Cn^{-\alpha-1} + O(n^{-\alpha-2}),$$

one finds that $q \geqq 2/5$ is sufficient for $\{n^{-\alpha}\}$ to multiply H^p into H^q. But the same device shows that $\{n^{-\alpha-1}\}$ is such a multiplier if $q \geqq 2/7$; hence so is $\{n^{-\alpha}\}$. Continuing this process, one proves the corollary for arbitrary $q > 0$.

The following result will be of use in proving the next two theorems. For a proof, see Zygmund [**5**, Vol. I, p. 214].

LEMMA 1. *If $\{a_n\}$ is a complex sequence such that $\sum |a_n|^2 = \infty$, then for some choice of signs $\epsilon_n = \pm 1$, the function $\sum_{n=0}^{\infty} \epsilon_n a_n z^n$ has a radial limit almost nowhere.*

PROOF OF THEOREM 2. It was shown in [**2**] that if $\{\lambda_n\}$ is a multiplier of H^p $(0 < p < \infty)$ into H^2, then

$$\sum_{n=1}^{N} n^{2/p} |\lambda_n|^2 = O(N^2). \tag{3}$$

(This condition is also sufficient if $0 < p \leqq 1$.) Given $\beta < 1/p - 1/2$, let $\lambda_n = \epsilon_n n^{-\beta}$, where the signs $\epsilon_n = \pm 1$ are as yet undetermined. Since the condition (3) is violated, $\{\lambda_n\}$ is not a multiplier of H^p into H^2. In other words, there exists a function $\sum a_n z^n$ in H^p such that $\sum |n^{-\beta} a_n|^2 = \infty$. Therefore, for a suitable choice of signs ϵ_n, the function $\sum \lambda_n a_n z^n$ has a radial limit almost nowhere, by Lemma 1. In particular, this function does not belong to H^q for any $q > 0$.

The following lemma will be needed in the proof of Theorem 3.

LEMMA 2. *If $1 < q < \infty$ and $1/q < \gamma < 1$, then $\{n^{-\gamma}\}$ is a multiplier of H^q into H^∞.*

PROOF. Let $\delta = 1 - \gamma$. Then $\delta < 1/q'$, so

$$g(z) = (1 - z)^{-\delta} = \sum_{n=0}^{\infty} b_n z^n$$

belongs to $H^{q'}$. Thus $\{b_n\}$ is a multiplier of H^q into H^∞ (see Caveny [**1**]). But

$$\Gamma(\delta) b_n = \Gamma(n + \delta)/n! = n^{-\gamma} + \nu_n,$$

where $\nu_n = O(n^{-\gamma-1})$. Since $\{\nu_n\} \in l^1$, it is clearly a multiplier of H^q into H^∞. Hence so is $\{n^{-\gamma}\}$.

PROOF OF THEOREM 3. We may assume $q < \infty$, since the case $q = \infty$ is covered by Theorem 1. Given $\beta < 1/2 - 1/q$, suppose that for every choice of signs $\epsilon_n = \pm 1$, $\{\epsilon_n n^{-\beta}\}$ is a multiplier of H^p into H^q. Let $\gamma = 1/2 - \beta$. By Lemma 2, $\{n^{-\gamma}\}$ multiplies H^q into H^∞, so $\{\epsilon_n n^{-\beta-\gamma}\} = \{\epsilon_n n^{-1/2}\}$ multiplies H^p into H^∞. Thus by the theorem of Caveny [1],

$$\sum_{n=1}^{\infty} \epsilon_n n^{-1/2} z^n \in H^{p'}$$

for every sign sequence $\{\epsilon_n\}$. But since $\{n^{-1/2}\} \notin l^2$, this contradicts Lemma 1. This proves the existence of a sign sequence $\{\epsilon_n\}$ such that $\{\epsilon_n n^{-\beta}\}$ is not a multiplier of H^p into H^q.

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