MULTIPLIERS OF H^1 AND HANKEL MATRICES

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1. **Introduction.** A function $f(z)$ analytic in the unit disc belongs to the class H^1 if

$$\|f\|_1 = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty.$$

A sequence $\lambda = \{\lambda(n)\}$ of complex numbers is said to be a multiplier of H^1 into the sequence space l^1 if $\lambda f = \{\lambda(n)\hat{f}(n)\} \in l^1$ for every $f(z) = \sum_0^\infty \hat{f}(n)z^n \in H^1$. The space of all such multipliers is denoted (H^1, l^1).

The only important known result about (H^1, l^1) is the inequality of Hardy [**5**, p. 236]: $\{1/(n+1)\} \in (H^1, l^1)$. Other similar multiplier spaces have been completely characterized: an elementary sufficient condition for (H^1, l^2), proved by Hardy and Littlewood [**4**], is also necessary, and the spaces (H^1, l^q) for $2 \leqq q \leqq \infty$ can be described similarly. Also (H^p, l^1) has been determined recently for $0<p<1$ by Duren and Shields [**3**], while (H^2, l^1) is trivial. (H^1, l^1) is a more interesting space, due to its equivalent formulations; perhaps in consequence it seems more difficult to determine. In this paper we give an alternate description of (H^1, l^1) in terms of Hankel matrices and use this matrix description to derive some necessary and some sufficient conditions.

2. **Observations and equivalences.** Observe initially that $\{\lambda(n)\} \in (H^1, l^1)$ if and only if $\{|\lambda(n)|\} \in (H^1, l^1)$; thus we may assume throughout that $\lambda(n) \geqq 0$. It follows from the Closed Graph Theorem that a multiplier is a bounded operator. Write $\|\lambda\|$ for its norm.

Let T_λ denote the Hankel matrix $(T_\lambda)_{ij} = \lambda(i+j)$ viewed as an operator on the Hilbert space l^2.

THEOREM 1. *$\lambda \in (H^1, l^1)$ if and only if T_λ is a bounded operator on l^2.*

PROOF. Write $\langle c, d\rangle$ for the inner product in l^2. Fix $c = \{c(n)\} \in l^2$ with $c(n) \geqq 0$. Then $f(z) = (\sum_0^\infty c(n)z^n)^2 \in H^1$ and $\|f\|_1 = \|c\|_2^2$. A calculation shows that $\langle T_\lambda c, c\rangle = \|\lambda f\|_1 \leqq \|\lambda\|\,\|f\|_1 = \|\lambda\|\,\|c\|_2^2$. Hence

$$\|T_\lambda\| = \sup\{\langle T_\lambda c, c\rangle : \|c\|_2 \leqq 1,\ c(n) \geqq 0\} \leqq \|\lambda\|.$$

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Conversely, fix $f\in H^1$ and write $f=gh$ with $\|f\|_1=\|g\|_2^2=\|h\|_2^2$. If $c(n)=|\hat{g}(n)|$ and $d(n)=|\hat{h}(n)|$, then $\|\lambda f\|_1\leqq\langle T_\lambda c,\ d\rangle\leqq\|T_\lambda\|\ \|f\|_1$. Thus $\|\lambda\|\leqq\|T_\lambda\|$.

If $\lambda(n)=1/(n+1)$, then T_λ is the Hilbert matrix. Thus the Hardy inequality is equivalent to the boundedness of the Hilbert matrix.

Another interesting reformulation was proved by Nehari [7]: T_λ is a bounded operator on l^2 if and only if there is a function $\phi\in L^\infty$ with $\hat{\phi}(n)=\lambda(n)$ for all $n\geqq 0$. (Here $\lambda(n)$ need not be nonnegative.)

3. **Necessary conditions.** Some of the following results are best stated in the language of mixed norm spaces. For $1\leqq r,\ s<\infty$, let $L^{r,s}$ be the space of sequences with finite norm

$$\|\lambda\|_{r,s}=\left(\sum_{m=0}^{\infty}\left(\sum_{n\in I(m)}|\lambda(n)|^r\right)^{s/r}\right)^{1/s}$$

where $I(m)=(2^{m-1},\ 2^m]$. If either index is infinite replace the corresponding sum by a supremum. For details see [1] and [6].

THEOREM 2. *If* $\lambda\in(H^1,\ l^1)$ *then* λ *satisfies* (N): $\sum_{j=0}^{\infty}(\sum_{i=j}^{j+k}\lambda(i))^2=O(k)$.

PROOF. Set $y_k=e_0+\cdots+e_k$ where $\{e_j\}_0^\infty$ is the standard basis of l^2. Then $\|y_k\|_2^2=k+1$ so that

$$\|T_\lambda y_k\|_2^2=\sum_{j=0}^{\infty}\left(\sum_{i=j}^{j+k}\lambda(i)\right)^2=O(k).$$

COROLLARY. *If* $\lambda\in(N)$ *then* $\lambda\in l^2$ *and* $\lambda\in L^{1,\infty}$.

We conjecture that (N) is sufficient as well as necessary. Examples may be found of sequences $\lambda\in l^2\cap L^{1,\infty}$ which fail to satisfy (N); thus the conditions of the corollary are not sufficient.

4. **Sufficient conditions.**

THEOREM 3. $L^{1,2}\subset(H^1,\ l^1)$.

PROOF. Fix $f\in H^1$. Let $n(m)$ be the index in $I(m)$ where $|\hat{f}(n)|$ is maximal. Then $E=\{n(m)\}$ is a Paley set so that [8] its characteristic function $\chi_E\in(H^1,\ l^2)$. Thus

$$\|\lambda f\|_1=\sum_{n=0}^{\infty}\lambda(n)|\hat{f}(n)|\ \leqq\sum_{m=1}^{\infty}|\hat{f}(n(m))|\sum_{I(m)}\lambda(n)\leqq\|\chi_E f\|_2\|\lambda\|_{1,2}.$$

It is interesting to compare the necessary conditions and sufficient conditions obtained so far: $L^{1,\infty}$ is necessary and $L^{1,2}$ sufficient; also l^2 is necessary and l^1 sufficient. All of the indexes are best possible, so any condition both necessary and sufficient must lie in between.

Two special cases provide evidence supporting the conjecture that (N) is sufficient. If λ is a lacunary sequence (say $\lambda(n)=0$ for all $n\neq 2^m$) then $\lambda\in L^{1,2}$ if and only if $\lambda\in l^2$; thus (N) is sufficient if the nonzero terms of λ are very sparse. The opposite extreme is when they are dense, in some sense. One possible interpretation is monotonicity.

THEOREM 4. *If* $\lambda(n)\downarrow 0$ *then* $\lambda\in(H^1, l^1)$ *if and only if* $\lambda(n)=O(1/n)$. *Thus* (N) *is necessary and sufficient.*

PROOF. Necessity is immediate, and sufficiency follows from the Hardy inequality.

This latter case admits a generalization for which (N) is again sufficient. It is generally correct that if one introduces blocks of zeros between the terms of a multiplier in a sufficiently regular manner the resulting sequence remains a multiplier. For example, for $r\geqq 1$ define $\lambda_r(n^r)=1/n$, $\lambda_r(m)=0$ for $m\neq n^r$. The Hilbert matrix corresponds to $r=1$. The λ_r for $r\geqq 1$ are again multipliers, although any sequence μ with $\mu(n^r)=(1/n)^\alpha$ for $\alpha<1$ is not. Such examples lead to consideration of sequences which are regular in the following sense: $\lambda(n)=0$ throughout $(2^{m-1}, 2^m]$ except for $\phi(m)$ terms all of size $\leqq\psi(m)$, equally spaced in the interval. (The equal spacing requirement may be relaxed enough that the sequences λ_r are included under Theorem 5.)

THEOREM 5. *If* $\lambda\in(N)$ *and if there is a constant* $c<1$ *with* $\psi(m)=O(c^m)$ *then* $\lambda\in(H^1, l^1)$.

PROOF. Since $\lambda\in(N)\subset L^{1,\infty}$ we may suppose that $\phi(m)\psi(m)\leqq 1$. We may further assume that $c>1/2$. Define $p(n)\equiv c^m$ in $I(m)$. We shall show that $\sum_{j=0}^{\infty} p(j)\lambda(j+n)=O(p(n))$; it follows from the Schur Lemma [2] that T_λ is bounded.

Suppose first that $n=2^k$. Then

$$\begin{aligned}\sum_{j=0}^{\infty} p(j)\lambda(j+2^k) \leqq \sum_{m=0}^{k} c^m\psi(k+1)\max\{1,\ 2^{m-1-k}\phi(k+1)\}&\\ +\sum_{m=k+1}^{\infty} c^m[\psi(m)\max\{1,\ (2^{m-1}-2^k)2^{1-m}\phi(m)\}&\\ +\psi(m+1)\max\{1,\ 2^{k+1-m}\phi(m+1)\}]&\\ \leqq c^{k-1}\sum_{m=0}^{k} c^m+2^{-k-1}\sum_{m=0}^{k} c^m 2^m+2\sum_{m=k+1}^{\infty} c^m+\sum_{m=k+1}^{\infty} c^m&\\ =O(c^k).&\end{aligned}$$

The general case follows from this by noting that the corresponding nonzero terms of λ are multiplied by smaller values of $p(n)$.

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