ON THE KERNEL FUNCTION FOR THE INTERSECTION OF TWO SIMPLY CONNECTED DOMAINS

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1. **Introduction.** Let D_1 and D_2 be bounded simply connected domains in the complex plane each containing the origin and let D be the component of $D_1 \cap D_2$ which contains the origin. It is clear that D is simply connected. Let $\{W_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ be complete orthonormal sets in the spaces $L^2(D_1)$ and $L^2(D_2)$ respectively (if G is a domain then $L^2(G)$ is the space of functions f analytic in G with $\iint_G |f|^2 < \infty$). In this paper we show that the set $\{W_n: n=1, 2, \cdots\}$ $\cup \{V_n: n=1, 2, \cdots\}$, with the domain restricted to D in each case, spans $L^2(D)$. This means that given functions f_1 and f_2 which map D_1 and D_2 conformally onto the disk $|z| < 1$ one can construct a function f which maps D conformally onto the disk. This can be done as follows. Obtain $\{W_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ from f_1 and f_2 [**3**, p. 247] then construct a complete orthonormal set $\{Q_n\}_{n=1}^{\infty}$ for $L^2(D)$. The Bergman kernel function $K(z, \zeta)$ for D is

$$K(z, \zeta) = \sum_{n=1}^{\infty} Q_n(z)\overline{Q_n(\zeta)}.$$

We may then choose f so that $f(0) = 0$ and $f'(z) = (\pi/K(0, 0))^{1/2} K(z, 0)$.

We observe that the result is clearly true in case the complements of D_1 and D_2 are closed domains. In this case the set $\{z^n: n=0, 1, \cdots\}$ spans each of $L^2(D_1)$, $L^2(D_2)$ and $L^2(D)$, [**2**] and [**3**, p. 254].

2. **Proof of the Theorem.** Let f_1, f_2 and f be functions which map D_1, D_2 and D respectively onto the disk $|z| < 1$ with $f_1(0) = f_2(0) = f(0) = 0$. Suppose $g \in L^2(D)$. Since $\{(n+1/\pi)^{1/2} f^n(z) f'(z)\}_{n=0}^{\infty}$ is a complete orthonormal set in $L^2(D)$ [**3**, p. 247] we may write

$$g(z) = \sum_{n=0}^{\infty} (n + 1/\pi)^{1/2} b_n f^n(z) f'(z), \qquad z \in D, \tag{1}$$

the series being absolutely and uniformly convergent on compact subsets of D and

$$\iint_D |g|^2 = \sum_{n=0}^{\infty} |b_n|^2 < \infty. \tag{2}$$

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Define

(3) $$g_\rho(z) = \sum_{n=0}^{\infty} (n + 1/\pi)^{1/2} \rho^n b_n f^n(z) f'(z), \qquad 0 < \rho < 1.$$

LEMMA 1. *There exists $g_\rho^{(1)}(z)$ analytic in D_1 and $g_\rho^{(2)}(z)$ analytic in D_2 such that*

(4) $$g_\rho(z) = g_\rho^{(1)}(z) + g_\rho^{(2)}(z) \quad \textit{when } z \in D.$$

PROOF. Let $\{r_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} r_n = 1$. For $0 < r < 1$, define

$$\begin{aligned} D_1^{(r)} &= f_1^{-1}(rf_1(z)), & z &\in D_1, \\ D_2^{(r)} &= f_2^{-1}(rf_2(z)), & z &\in D_2, \\ D^{(r)} &= f^{-1}(rf(z)), & z &\in D, \end{aligned}$$

and let $C_1^{(r)}$, $C_2^{(r)}$ and $C^{(r)}$ be the corresponding boundaries. Let D_r be the component of $D_1^{(r)} \cap D_2^{(r)}$ which contains the origin and let $\{R_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} R_n = 1$ and $D^{(R_n)} \supset \overline{D}_{r_n}$, $n = 1, 2, \cdots$. We thus have a sequence $\{D_{r_n}\}$ of domains such that $D_{r_{n+1}} \supset D_{r_n}$, $\bigcup_{n=1}^{\infty} D_{r_n} = D$ and a sequence $\{C^{(R_n)}\}_{n=1}^{\infty}$ of closed curves contained in D such that $\overline{D}_{r_n}$ is contained in the interior of $C^{(R_n)}$ and for $z \in D_{r_n}$ and $\zeta \in C^{(R_n)}$,

(5) $$\min |\zeta - z| = M_n > 0, \qquad n = 1, 2, \cdots.$$

We now wish to find $\{\Gamma_n^{(1)}\}$ and $\{\Gamma_n^{(2)}\}$ such that $\Gamma_n^{(1)} \cup \Gamma_n^{(2)} = C^{(R_n)}$ and such that $\Gamma_n^{(1)} \cap D_1^{(r_n)}$ and $\Gamma_n^{(2)} \cap D_2^{(r_n)}$ are empty, $n = 1, 2, \cdots$. This is accomplished by setting $\Gamma_n^{(1)} = C^{(R_n)} \cap D_2^{(r_n)}$ and $\Gamma_n^{(2)} = C^{(R_n)} - \Gamma_n^{(1)}$.

For $j = 1, 2$, $m \geqq n = 1, 2, \cdots$ and $z \in D_j^{(r_n)}$, define

(6) $$h_{m,n}^{(j)}(z) = \frac{1}{2\pi i} \int_{\Gamma_m^{(j)}} \frac{g_\rho(\zeta)}{\zeta - z} \, d\zeta.$$

Then when $z \in D_{r_n}$, $m \geqq n$ we have

(7) $$g_\rho(z) = h_{m,n}^{(1)}(z) + h_{m,n}^{(2)}(z).$$

Further, if $z \in D_j^{(r_n)} \cap D^{(r_{n'})}$ with $m \geqq n$ and $m \geqq n'$ then

(8) $$h_{m,n}^{(j)}(z) = h_{m,n'}^{(j)}(z).$$

We now fix n and show that $\{h_{m,n}^{(j)}\}_{m=n}^{\infty}$ is a normal family in

$D_j^{(r_n)}$, $j=1, 2$. Since the series (1) converges uniformly in $\bar{D}^{(R_n)}$, we may write

$$(9) \qquad h_{m,n}^{(j)}(z) = \sum_{k=0}^{\infty} ((k+1)/\pi)^{1/2} \rho^k b_k a_{m,n}^{(j)}(z; k), \qquad z \in D_j^{(r_n)},$$

$j=1, 2$, where

$$\begin{aligned} a_{m,n}^{(j)}(z; k) &= \frac{1}{2\pi i} \int_{\Gamma_n^{(j)}} \frac{f^k(\zeta) f'(\zeta)}{\zeta - z} d\zeta. \\ &= \frac{1}{2\pi i} \int_{f(\Gamma_m^{(j)})} \frac{w^k}{f^{-1}(w) - z} dw. \end{aligned}$$

But $w \in f(\Gamma_m^{(j)})$ implies $|w| = R_m < 1$ so

$$(10) \qquad \left| a_{m,n}^{(j)}(z; k) \right| \leqq \frac{1}{2\pi} \int_{f(\Gamma_m^j)} \frac{|dw|}{M_n} \leqq \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{M_n} = \frac{1}{M_n} \, .$$

Now (9) and (10) imply

$$(11) \qquad \left| h_{m,n}^{(j)}(z) \right| \leqq \sum_{k=0}^{\infty} ((k+1)/\pi)^{1/2} \rho^k \left| b_k \right| \cdot \frac{1}{M_n}, \qquad z \in D_j^{(r_n)}.$$

The series (11) converges by (2) and the fact that $\rho < 1$. Hence for fixed n and $j=1, 2$, the family $\{h_{m,n}^{(j)}\}_{m=n}^{\infty}$ is uniformly bounded in $D_j^{(r_n)}$ and is therefore a normal family. Let I_1^1 be a subset of the positive integers such that $\{h_{m,1}^{(1)}: m \in I_1^1\}$ converges to a function, say $h_1^{(1)}$, analytic in $D_1^{(r_1)}$. Let $I_1^2 \subset I_1^1$ be such that $\{h_{m,1}^{(2)}: m \in I_1^2\}$ converges to $h_1^{(2)}$ analytic in $D_2^{(r_1)}$. Continuing in the same manner choose $I_n^2 \subset I_n^1$ so that $\{h_{m,n}^{(2)}: m \in I_n^2\}$ converges to $h_n^{(2)}$ in $D_2^{(r_n)}$ and $I_n^1 \subset I_{n-1}^2$ so that $\{h_{m,n}^{(1)}: m \in I_n^1\}$ converges to $h_n^{(1)}$ in $D_1^{(r_n)}$. Using (7) and (8), we then conclude

$$(12) \qquad \begin{aligned} g_\rho(z) &= h_n^{(1)}(z) + h_n^{(2)}(z) \quad \text{if } z \in D_{r_n}, \\ h_n^{(j)}(z) &= h_{n'}^{(j)}(z) \quad \text{if } z \in D_j^{(r_n)} \cap D_j^{(r_{n'})}. \end{aligned}$$

Now define $g_\rho^{(j)}$ in D by $g_\rho^{(j)}(z) = h_n^{(j)}(z)$ if $z \in D_j^{(r_n)}$. Then (12) implies that $g_\rho^{(j)}$ is well defined and analytic in D_j ($j=1, 2$) and

$$g_\rho(z) = g_\rho^{(1)}(z) + g_\rho^{(2)}(z), \qquad z \in D.$$

LEMMA 2. *We may write*

$$(13) \qquad g_\rho^{(j)}(z) = \sum_{k=0}^{\infty} a_k^{(j)} [f_j(z)]^k,$$

the series being absolutely and uniformly convergent on compact subsets of D_j, $j=1, 2$.

PROOF. The function $g_\rho^{(j)}(f_j^{-1}(w))$ is analytic in $|w|<1$ so $g^{(j)}(f_j^{-1}(w)) = \sum_{k=0}^{\infty} a_k^{(j)} w^k$ and setting $z=f_j^{-1}(w)$, (13) follows.

Now let $\{W_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ be complete orthonormal sets in $L^2(D_1)$ and $L^2(D_2)$ respectively. Let $\{Q_n\}_{n=1}^{\infty}$ be an orthonormal set in $L^2(D)$ obtained by choosing a maximal linearly independent set from $\{W_n\}\cup\{V_n\}$ and orthonormalizing it.

Since $[f_j(z)]$, $j=1, 2$ is bounded, $f_j^n \in L^2(D_j)$, $n=0, 1, \cdots$, so we may write

$$
(14)\qquad
\begin{aligned}
f_1^n(z) &= \sum_{k=1}^{\infty} a_{k,n} Q_k(z),\\
f_2^n(z) &= \sum_{k=1}^{\infty} b_{k,n} Q_k(z), \qquad z \in D,
\end{aligned}
$$

the series converging uniformly and absolutely on compact subsets of D. Hence

$$
(15)\qquad
\begin{aligned}
g_\rho(z) &= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{k,n} Q_k(z) + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} b_{k,n} Q_k(z)\\
&= \sum_{k=1}^{\infty} P_k Q_k(z), \qquad z \in D.
\end{aligned}
$$

The rearrangement is possible in (15) since the series (14) converge absolutely on compact subsets of D.

Now let $\epsilon>0$ choose ρ so that

$$\iint_D |g - g_\rho|^2 = \sum_{n=0}^{\infty} (1-\rho^n)^2 |b_n|^2 < \epsilon.$$

It is known [1, p. 2] that $\iint_D |g - \sum_{n=1}^{\infty} c_n Q_n|^2$ is a minimum when

$$c_n = d_n = \iint_D g(z)\overline{Q_n(z)}\,dxdy, \qquad n = 1, 2, \cdots.$$

Hence we have

$$
\begin{aligned}
\iint_D \left| g - \sum_{n=1}^{\infty} d_n Q_n \right|^2 &\leq \iint_D \left| g - \sum_{n=1}^{\infty} P_n Q_n \right|^2\\
&= \iint_D |g - g_\rho|^2 < \epsilon.
\end{aligned}
$$

This implies $g(z)=\sum_{n=1}^{\infty} d_n Q_n(z)$, $z\in D$ and that $\sum_{n=1}^{\infty} |d_n|^2 = \iint_D |g(z)|^2\, dxdy < \infty$. This proves the following theorem.

THEOREM. *The set* $\{Q_n\}_{n=1}^{\infty}$ *is complete in* $L^2(D)$.

3. Discussion. Two interesting questions remain open. Given $g\in L^2(D)$ it would be desirable to obtain $g^{(1)}(z)$ and $g^{(2)}(z)$ analytic in D_1 and D_2 respectively so that

$$g(z) = g^{(1)}(z) + g^{(2)}(z), \qquad z \in D.$$

However the present method does not yield this result. It seems necessary to use $g_\rho(z)$ in order to obtain normal families. Then $g_\rho(z)$ can be written in the form (4).

A second open question is whether we may require $g_\rho^{(j)}\in L^2(D_j)$, $j=1, 2$ in (4).

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