A NOTE ABOUT WIENER-HOPF SETS

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DEFINITION. A Wiener-Hopf set is a subset S of the real line R such that

(i) S^+ and S^- are not empty,

(ii) $S^+ + S^- \subset S$,

where $S^+ \cup S^0 \cup S^-$ is the usual decomposition of S into positive, negative and null parts.

Let us explain this definition: Considering a probability measure μ on R, the Wiener-Hopf decomposition of μ is given by:

$$\epsilon_0 - \mu = (1 - \alpha)(\epsilon_0 - \mu^-) * (\epsilon_0 - \mu^+) \tag{1}$$

where the star indicates convolution, ϵ_0 is the unit mass at 0, $0 \leqq \alpha \leqq 1$, and μ^+ and μ^- are positive measures of mass not larger than one concentrated on $R^+ = \{x: x > 0\}$ and $R^- = \{x: x < 0\}$. The measure μ^- (resp. μ^+) has a probabilistic interpretation as a distribution of the first visit to R^- (resp. R^+) of the random walk $S_n = X_1 + \cdots + X_n$ where $X_1, \cdots, X_n$ are independent random variables with the same distribution μ; the number α is the probability that the first visit to $R^+ \cup \{0\}$ (or $R^- \cup \{0\}$) is to 0. See Feller [1] for details.

(1) can be rewritten:

$$\mu = \alpha\epsilon_0 + (1 - \alpha)(\mu^+ + \mu^- - \mu^+ * \mu^-) \tag{2}$$

This formula gives us two examples of Wiener-Hopf sets.

PROPOSITION 1. *If $\mu(R^+)$ and $\mu(R^-)$ are positive the support of $\alpha\epsilon_0 + \mu^+ + \mu^-$ is a Wiener-Hopf set.*

PROOF. Let us denote by $S(\nu)$ the support of any measure ν, that is to say the smallest closed subset of R carrying the whole mass of ν. If $S^+ \cup S^0 \cup S^- = S = S(\alpha\epsilon_0 + \mu^+ + \mu^-)$ is the decomposition on R^+, 0, R^- of S, we have $S(\mu^+) \supset S^+$ and $S(\mu^-) \supset S^-$. μ positive implies, by (2), $S \supset S(\mu^+ * \mu^-)$. But $S(\mu^+ * \mu^-) \supset S(\mu^+) + S(\mu^-)$. Then $S^+ + S^- \subset S$. Since by hypothesis S^+ and S^- are not empty, S is a Wiener-Hopf set.

Let us denote by ν_a the atomic part of any measure ν.

PROPOSITION 2. *If $\mu_a(R^+)$ and $\mu_a(R^-)$ are positive the set of atoms of $\alpha\epsilon_0 + \mu^+ + \mu^-$ is a Wiener-Hopf set.*

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PROOF. (2) implies $\mu_a = \alpha\epsilon_0 + (1-\alpha)(\mu_a^+ + \mu_a^- - \mu_a^+ * \mu_a^-)$ and the proof goes in a similar way to that in Proposition 1.

We are led to consider Wiener-Hopf sets from the following observed fact: Suppose $S(\mu)$ is a set of integers. Then $S(\alpha\epsilon_0+\mu^+ +\mu^-)$ has no holes, that is to say, it is an interval (not necessarily bounded) of integers. A similar property holds when $S(\mu)$ is not concentrated on lattice points.

The aim of this note is to provide the proof of these results in the following theorem:

THEOREM. *If S is a Wiener-Hopf set, let $\rho = \inf \{x: x \in S^+\}$. Either $\rho > 0$ and S/ρ is an interval of integers, or $\rho = 0$ and the closure $\overline{S}$ of S is an interval of R.*

Let us introduce two useful notations: $[x, y]_Z$ is the interval of integers with end points x and y. When $x<0$ and $y>0$, let $A(x, y) = [x, y] \cap \{mx+ny: m, n \text{ nonnegative integers}, m+n>0\}$. We need now three lemmas:

LEMMA 1. *If p and q are positive integers such that $(p, q)=1$, then $A(-p, q) = [-p, q]_Z$.*

PROOF. Easy, using Bezout identity.

LEMMA 2. *If x/y is irrational, $A(x, y)$ is dense in $[x, y]$.*

PROOF. Easy, using the fact that multiples of a positive irrational number, taken modulo 1, are dense in $[0, 1]$.

LEMMA 3. *$x \in S^-$ and $y \in S^+$ imply $A(x, y) \subset S$.*

PROOF. We use induction on $N = m+n$. Suppose that $mx+ny \in S$ for any m and n such that $x \leqq mx+ny \leqq y$ and $m+n<N$. This is true for $N=2$. If $x \leqq mx+ny \leqq y$ and $m+n=N \geqq 2$ then $m \geqq 1$, $n \geqq 1$, $(m-1)x+ny \geqq 0$ and $mx+(n-1)y \leqq 0$. Clearly, at least one of these two numbers (say, the first) is in $[x, y]$ and by the induction hypothesis, is in S. Hence either $(m-1)x+ny \in S^0$ and $mx+ny = x \in S$, or $(m-1)x+ny \in S^+$ and $mx+ny \in S^- + S^+ \subset S$.

PROOF OF THE THEOREM. Suppose $\rho>0$. We claim first $\rho \in S^+$. If not, there exists a strictly decreasing sequence (y_n) in S such that $y_n \underset{n\infty}{\rightarrow} \rho$. Since S^- is not empty choose $x \in S^-$. If there existed n such that x/y_n is irrational, by Lemmas 2 and 3 ρ would be 0. Hence x/y_n is rational for any n, and there exist $\lambda_n > 0$ and positive integers p_n and q_n such that $(p_n, q_n)=1$ and $x = -\lambda_n p_n$ and $y_n = \lambda_n q_n$. But the $x/y_n = p_n/q_n$ being distinct, the sequence q_n is not bounded, and there exists a sequence n_k of integers such that $q_{n_k} \underset{k\infty}{\rightarrow} \infty$ and $\lambda_{n_k} \rightarrow 0$. By

Lemma 1, $A(x, y_{n_k}) = \lambda_{n_k}[-p_{n_k}, q_{n_k}]_Z$ and, by Lemma 3, $\rho = 0$. Hence $\rho > 0$ implies $\rho \in S^+$.

Choose $x \in S^-$. We show that $x = -n\rho$ where

$$n + 1 = \min \{k: x + k\rho \in S^+\}.$$

By Lemma 3, $x + n\rho \in S^- \cup S^0$. If $x + n\rho \in S^-$, then $x + (n+1)\rho < \rho$, a contradiction. Hence $x + n\rho \in S^0$. Furthermore $0 \in S$, $A(x, \rho) = \rho[-n, 1]_Z \subset S$ and $-\rho = \max \{x: x \in S^-\}$. Using the same proof one can show that every element of S^+ is a multiple of ρ, and S/ρ is an interval of Z.

Suppose $\rho = 0$. Let $x \in S^-$, $\epsilon > 0$ and t be such that $x < t < 0$. We claim that $(t - \epsilon, t) \cap S^-$ is not empty. There exists a strictly decreasing sequence (y_n) in S^+ such that $y_n \underset{n\infty}{\to} 0$. Define k_n, nonnegative integer, and ϵ_n, by $t - x = k_n y_n + \epsilon_n$ with $0 \leqq \epsilon_n < y_n$. If y_n is such that $y_n < \epsilon$ then $x + k_n y_n \in (t - \epsilon, t)$. But by Lemma 3, $x + k_n y_n \in S^-$. Hence S^- is dense in $[x, 0]$. The proof for S^+ goes in the same way and the closure of S is an interval of R.

Bibliography

1. W. K. Feller, *An introduction to probability theory and applications*, Vol. II, Wiley, New York, 1966.

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