THE STRUCTURE OF $B[c]$ AND EXTENSIONS OF THE CONCEPT OF CONULL MATRIX

H. I. BROWN,[1] D. R. KERR[1] AND H. H. STRATTON[2]

Introduction. The algebra Γ of conservative matrices is partitioned by the ideal Ψ of conull matrices and the set of coregular matrices. This paper is concerned with the problem of extending the concept of conullity from Γ to the algebra $B[c]$ of bounded linear operators on the space c of convergent sequences. Since Ψ can be realized as the kernel of a scalar homomorphism χ on Γ (i.e., χ is a linear functional on Γ so that $\chi(AB)=\chi(A)\chi(B)$ for A, $B\in\Gamma$), one possibility is to consider extensions of χ in hopes that their kernels will be the natural extensions of Ψ. Wilansky [2, p. 250, Question 3] considers two such extensions, χ and ρ, and observes that they both fail to be a scalar homomorphism on $B[c]$. By investigating the algebraic structure of $B[c]$ it is seen that, in a sense, Wilansky's extensions are the only ones possible. For example, in §1 it is proved that the only subalgebras of $B[c]$ which properly contain Ψ are Γ, $\rho_\perp$ (the kernel of ρ), Ω (the domain of ρ), and $B[c]$. In §3 it is proved that χ is the only nonzero scalar homomorphism on Γ, that ρ is the only nonzero scalar homomorphism on Ω, and that there are no nonzero scalar homomorphisms on $B[c]$. From these facts we can conclude that ρ is the only extension of χ to a scalar homomorphism on a subalgebra of $B[c]$. Therefore, if we are to extend the concept of conullity by means of scalar homomorphisms, then $\rho_\perp$ becomes the natural definition for conullity in $B[c]$.

Let M designate the multiplicative operators in $B[c]$ (defined below). Unlike the conull-coregular dichotomy in Γ, the dichotomy of multiplicative and nonmultiplicative matrices cannot be effected by a scalar homomorphism. We show that this behavior is retained in subalgebras of $B[c]$ containing $M\cap\Gamma$. For example, in §2 it is seen that the only subalgebras of $B[c]$ which properly contain $M\cap\Gamma$ are Γ, $M\cap\Omega_0$ (defined below), $M\cap\Omega$, Ω, M, and $B[c]$. Moreover, in §3 it is proved that ρ and χ are the only nonzero scalar homomorphisms on any of the above subalgebras of M. Hence no one of these algebras can be realized as the kernel of a scalar homomorphism on a larger algebra.

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Definitions, notation, and background. c is the Banach space of convergent sequences x with $\|x\| = \sup |x_i|$. For $x \in c$, $\lim x$ means $\lim_i x_i$. Let e, e^k for $k \in I^+$ be those elements of c defined respectively by $e_i = 1$ for each $i \in I^+$ and $e_i^k = \delta_{ik}$ where δ_{ik} is the Kronecker delta. $B[c]$ is the Banach algebra of bounded linear operators on c with $\|T\| = \sup\{\|Tx\| : x \in c \text{ and } \|x\| \leqq 1\}$. For each $T \in B[c]$, $\chi(T) = \lim Te - \sum_k \lim Te^k$ and $\chi_i(T) = (Te)_i - \sum_k (Te^k)_i$ for each $i \in I^+$, [2, p. 241]. Let

$$\Omega = \left\{T \in B[c] \colon \lim_i \chi_i(T) \text{ exists,}\right\}$$

$$\Omega_0 = \left\{T \in B[c] \colon \lim_i \chi_i(T) = 0\right\},$$

$$\Gamma = \{T \in B[c] \colon T \text{ is a matrix}\},$$

$$\Psi = \{T \in \Gamma \colon \chi(T) = 0\},$$

$$M = \{T \in B[c] \colon \text{For some scalar } \alpha, \lim Tx = \alpha \lim x \text{ for each } x \in c\}.$$

Γ has the equivalent definition [2, p. 241]

$$\{T \in B[c] \colon \chi_i(T) = 0 \text{ for each } i \in I^+\}.$$

We can write each $T \in B[c]$ as follows:

$$Tx = (\lim x)v + Bx \qquad \text{for } x \in c,$$

where $v = \{\chi_i(T)\}$, and B is the matrix obtained by restricting T to those elements of c which converge to 0. This relationship between T, v, and B is denoted by $T \sim (v, B)$. Further discussion of this representation is to be found in [1].

For $T \in \Omega$ define $\rho(T) = \chi(B)$, where $T \sim (v, B)$. Note that since $T \in \Omega$, $B \in \Gamma$, and so the definition makes sense.

For $T \in B[c]$, $[T]$ is the matrix representation of the second adjoint of T. The entries of $[T]$ will be designated by the upper case letters T_{ij} and are defined as follows: $T_{11} = \chi(T)$, $T_{i1} = \chi_{i-1}(T)$ for $i > 1$, $T_{1j} = \lim Te^{j-1}$ for $j > 1$, and $T_{ij} = b_{i-1,j-1}$ for $i > 1$ and $j > 1$, where $T \sim (v, B)$. $[T]$ is related to T by the equation $Tx = J^{-1}[T]Jx$ for each $x \in c$, where $Jx = \{\lim x, x_1, x_2, \cdots\}$. The representation $[T]$ has the advantage that it displays many salient properties of T, and will be used extensively in what follows.

For $T \in \Omega$ define $V(T) \in \Omega$ by $V(T)_{11} = \lim \chi_i(T)$, $V(T)_{i1} = \chi_{i-1}(T)$ for $i > 1$, $V(T)_{ij} = 0$ otherwise (where $V(T)_{ij}$ are the entries of $[V(T)]$), and define $B(T) \in \Gamma$ by $B(T)_{11} = \rho(T)$, $B(T)_{i1} = 0$ for $i > 1$, $B(T)_{ij} = T_{ij}$ otherwise. Note that $T = V(T) + B(T)$.

The following conventions are made. E always represents the operator $(e, 0)$. Λ always designates a subalgebra of $B[c]$, and every algebra mentioned is assumed to be a subalgebra of $B[c]$. All undesignated entries in a matrix are assumed to be zero. "Scalar homomorphism" means "nonzero scalar homomorphism."

Further properties of χ, ρ, $\chi_\perp$, Ω, $[T]$, (v, B), etc. are developed in [1].

1. **Algebras containing** Ψ. Results of this section establish that the only subalgebras of $B[c]$ which contain Ψ are Ψ, Γ, $\rho_\perp$, Ω, and $B[c]$. We begin with a technical lemma. Let m denote the Banach space of bounded sequences.

1.1. LEMMA. *If $z\in m$ and $x\in m-c$, then there exists $R\in M\cap\Psi$ so that $z=Rx$.*

PROOF. Let α and β be distinct cluster points of x with $\beta\neq 0$. Let $\{n(j)\}$ and $\{k(j)\}$ be disjoint strictly increasing subsequences of I^+ so that for $j\in I^+$, $x_{n(j)}\neq 0$ and $x_{k(j)}\neq\beta$, and so that $\lim_j x_{k(j)}=\alpha$ and $\lim_j x_{n(j)}=\beta$. Define the subsequence $\{m(j)\}$ of I^+ by $m(2j-1)=n(j)$ and $m(2j)=k(j)$.

We now define five matrices whose product is the desired matrix R.

$$V\colon\ v_{j,m(j)} = 1,$$

$$U\colon\ u_{2j-1,2j-1} = 1/(Vx)_{2j-1}, \qquad u_{2j,2j} = 1/\beta,$$

$$W\colon w_{jj} = 1, \qquad w_{2j,2j+1} = w_{2j-1,2j+1} = -1,$$

$$P\colon\ p_{2j,2j} = 1/((x_{k(j)}/\beta) - 1), \qquad p_{2j-1,2j-1} = 1/((\alpha/\beta) - 1),$$

$$Q\colon\ q_{jj} = -q_{j,j+1} = (-1)^j z_j.$$

Obviously V, U, W, P, and Q are all in M. Moreover, Q is in Ψ and Ψ is an ideal in Γ. Therefore, $R=Q\ P\ W\ U\ V\in M\cap\Psi$. That $z=Rx$ is a straightforward computation.

1.2. LEMMA. *No proper algebra contains $\chi_\perp$.*

PROOF. Since χ is linear, $\chi_\perp$ is a maximal subspace of $B[c]$. The result follows since $\chi_\perp$ is not an algebra [2, Lemma 3].

1.3. THEOREM. *$B[c]$ is the only algebra containing Ψ and some element not in Ω.*

PROOF. Let $\Lambda\supset\Psi$ and let $S\in\Lambda-\Omega$. By Lemma 1.2 it suffices to show that $\Lambda\supset\chi_\perp$. By Lemma 1.1, given $T\in\chi_\perp$ we may choose $R\in M\cap\Psi$ so that $\{\chi_i(T)\}=R\{\chi_i(S)\}$. Then $T-RS\in\Psi$ since the first column of $[T-RS]$ is zero. Hence $T=T-RS+RS\in\Lambda$, and so Λ must be $B[c]$.

Let B_c be the set of $T\in B[c]$ such that all columns of $[T]$ are zero except the first which is in c, and let B_0 be the set of those T in B_c whose first column converges to zero.

1.4. LEMMA. *Any algebra containing Ψ and a nonmatrix contains B_c.*

PROOF. Let $\Lambda\supset\Psi$ and let $S\in\Lambda-\Gamma$. If $T\in B_c$, then

$$T - \left(\lim_i T_{i1}\right)E \in B_0.$$

Hence to show that $\Lambda\supset B_c$ it suffices to show that $E\in\Lambda$ and that $\Lambda\supset B_0$.

Since $S\notin\Gamma$, $\chi_k(S)\neq 0$ for some $k\in I^+$. Define $V\in\Psi$ by $v_{kk}=1/\chi_k(S)$. Clearly $VS\in\Lambda$. Define $P\in\Psi$ by $p_{kj}=(Se^j)_k/\chi_k(S)$ for each $j\in I^+$. Then letting $F=VS-P$ we see that $F_{k+1,1}=1$. Thus, by defining $G\in\Psi$ by $G_{i,k+1}=1$ for $i\in I^+$, we establish that $E=GF\in\Lambda$.

Finally, if $T\in B_0$ define $R\in\Psi$ by $r_{ii}=T_{i1}$ for $i\in I^+$. Then $T=RE\in\Lambda$.

1.5. THEOREM. *The only algebras in Ω which contain Ψ and a nonmatrix are Ω and $\rho_\perp$.*

PROOF. That Ω is an algebra can be verified by computing $[S]\,[T]$ for S, $T\in\Omega$. To show that $\rho_\perp$ is an algebra we need only show that $ST\in\rho_\perp$ whenever S, $T\in\rho_\perp$. If $T\in\Omega$, then $[T]\in\Gamma$. Since χ is multiplicative on Γ and since $[ST]=[S]\,[T]$, it remains to show that $\rho(T)=\chi([T])$. But if $T\sim(v, B)$, *then*

$$\rho(T) = \chi(B) = \lim_n \sum_j b_{nj} - \sum_j \lim_n b_{nj} = \lim_n\left(\chi_n(T) + \sum_j b_{nj}\right)$$
$$- \left(\lim_n \chi_n(T) + \sum_j \lim_n b_{nj}\right) = \chi([T]).$$

Thus, $\rho_\perp$ is an algebra.

Now suppose that $\Omega\supset\Lambda\supset\Psi$ and $\Lambda-\Gamma\neq\emptyset$. Then by Lemma 1.4, $\Lambda\supset B_c$. Either $\Lambda-\rho_\perp\neq\emptyset$ or $\Lambda\subset\rho_\perp$.

In the first case, there exists $T\in\Lambda-\rho_\perp$. Since $T\in\Omega$, $T=V(T)+B(T)$ with $B(T)\in\Lambda$. (Indeed, $T\in\Lambda$ and $V(T)\in B_c\subset\Lambda$.) Thus if $S\in\Omega$, then $S=V(S)+B(S)$ with $V(S)\in\Lambda$ and $B(S)\in\Lambda$ because $B(S)-(\rho(S)/\rho(T))B(T)\in\Psi\subset\Lambda$. Hence $S\in\Lambda$, and $\Lambda=\Omega$.

In the second case, if $\Lambda\subset\rho_\perp$, then for $S\in\rho_\perp$, $B(S)\in\Psi\subset\Lambda$, and so $S\in\Lambda$. It follows that $\Lambda=\rho_\perp$. This completes the proof.

Noticing that there are no algebras between Γ and Ψ we now see that the only subalgebras which contain Ψ are Ψ, Γ, $\rho_\perp$, Ω, and $B[c]$.

2. **Algebras containing** $M\cap\Gamma$. It is not difficult to observe that $T\in M$ if and only if $\lim Te^i=0$ for each $i\in I^+$, and hence that M is an algebra. Using arguments similar to those in §1 we establish that the only algebras containing $M\cap\Gamma$ are $M\cap\Gamma$, Γ, $M\cap\Omega_0$, $M\cap\Omega$, Ω, M, and $B[c]$. It will be seen from the next section that no one of these algebras can be realized as the kernel of a scalar homomorphism on a larger algebra.

2.1. LEMMA. *If an algebra contains $M\cap\Psi$ and a nonmultiplicative operator, then it contains Ψ.*

PROOF. Let Λ be such an algebra and let $S\in\Lambda-M$. Then $\lim Se^k =b\neq 0$ for some $k\in I^+$. Let $T\in\Psi$ and $a_i=\lim Te^i$ for each $i\in I^+$. Define $V\in M\cap\Psi$ by $v_{kj}=a_j/b$ for each j. The first row of $[SV]$ is the same as the first row of $[T]$. Therefore, $T-SV\in M\cap\Psi$ so that $T=T-SV+SV\in\Lambda$.

2.2. THEOREM. *The only algebras containing $M\cap\Gamma$ and some element not in Ω are M and $B[c]$.*

PROOF. Let Λ be such an algebra. If $\Lambda-M\neq\emptyset$, then, by Lemma 2.1, $\Lambda\supset\Psi$. Hence by Theorem 1.3, Λ must be $B[c]$. On the other hand, suppose $\Lambda\subset M$. Let $T\in M$ and $S\in\Lambda-\Omega$. Using Lemma 1.1 choose $R\in M\cap\Psi\subset\Lambda$ so that $\{\chi_i(T)\}=R\{\chi_i(S)\}$. Then $RS\in\Lambda$ and $T-RS\in M\cap\Gamma$. Therefore, $T=T-RS+RS\in\Lambda$.

2.3. LEMMA. *If an algebra contains $M\cap\Gamma$ and a nonmultiplicative operator, then it contains the elements of Γ whose columns are constant sequences.*

PROOF. Let $\Lambda\supset M\cap\Gamma$ and let $S'\in\Lambda-M$. Then for some $k\in I^+$, $\lim S'e^k\neq 0$, and so for at most finitely many i, say $i(1),\cdots,i(n)$, $(S'e^k)_{i(l)}=0$. Define $U\in M\cap\Gamma$ by $\mu_{i(l),k}=1$ for $1\leqq l\leqq n$. Then $S=U+S'\in\Lambda$ and $(Se^k)_i\neq 0$ for each $i\in I^+$. Define $P\in M\cap\Gamma$ by $p_{ii}=1/(Se^k)_i$ for each $i\in I^+$. Then $PS\in\Lambda$. Finally, define $T\in M\cap\Gamma$ by $t_{kj}=\alpha_j$, where $\sum|\alpha_j|<\infty$. Then TPS is the matrix whose jth column has the constant value α_j and $TPS\in\Lambda$.

2.4. THEOREM. *There are no algebras between Γ and $M\cap\Gamma$.*

PROOF. Let $\Gamma\supset\Lambda\supset M\cap\Gamma$. If $\Lambda\neq M\cap\Gamma$, then, by Lemma 2.1, $\Lambda\supset\Psi$ and $\Lambda\neq\Psi$. But Ψ is a maximal subalgebra of Γ, and so we must have $\Lambda=\Gamma$.

2.5. THEOREM. *Any algebra containing* $M\cap\Gamma$ *and a nonmatrix contains* $M\cap\Omega_0$.

PROOF. Let Λ be such an algebra. As in the proof of Lemma 1.4, there exists $k\in I^+$ so that $F\in\Lambda$ where F is defined by $F_{k+1,1}=1$. Now let $S\in B_0$. Define $S'\in M\cap\Gamma$ by $S'_{i,k+1}=S_{i1}$. Then $S=S'F\in\Lambda$ and so $\Lambda\supset B_0$.

Now let $T\in M\cap\Omega_0$. As in Theorem 1.5, $T=V(T)+B(T)$ where $V(T)\in B_0$ and $B(T)\in M\cap\Gamma$. Hence $T\in\Lambda$.

2.6. THEOREM. *Any algebra containing* $M\cap\Gamma$ *and some nonmatrix not in* $M\cap\Omega_0$ *contains* $M\cap\Omega$.

PROOF. If Λ is such an algebra, then Theorem 2.5 implies $\Lambda\supset B_0$. Thus if $T\in M\cap\Omega$, then $V(T)\in B_c$ and $B(T)\in M\cap\Gamma$. Since B_0 is a maximal subspace of B_c, it suffices to establish the existence of some $S\in\Lambda\cap(B_c-B_0)$.

If $\Lambda-\Omega\neq\emptyset$, the theorem follows from Theorem 2.2, and if $\Lambda-M\neq\emptyset$ then, by Lemmas 2.1 and 1.4, $\Lambda\supset B_c$. Therefore, it suffices to consider the case $M\cap\Omega\supset\Lambda$.

Let $S'\in\Lambda-M\cap\Omega_0$. Then $B(S')\in M\cap\Gamma$ and so $V(S')\in\Lambda$. Since $S'\notin\Omega_0$, $V(S')\notin B_0$ and so $V(S')$ is the desired S.

2.7. THEOREM. *Any algebra containing* $M\cap\Gamma$ *and a nonmultiplicative operator is* Γ *or contains* Ω.

PROOF. By Lemma 2.3, if Λ is such an algebra, then Λ contains those elements of Γ whose columns are constant sequences. If $\Lambda\subset\Gamma$ then, by Theorem 2.4, $\Lambda=\Gamma$. If $\Lambda\not\subset\Gamma$ then, by Lemmas 2.1 and 1.4, $\Lambda\supset B_c$ and so, by Theorem 2.6, $\Lambda\supset M\cap\Omega$.

Now let $T\in\Omega$. Define $A\in\Lambda$ by $A_{i,j+1}=\lim Te^j$ for $j\in I^+$ and $i>1$. Then $T-A\in M\cap\Omega\subset\Lambda$ and so $T\in\Lambda$.

3. **Scalar homomorphisms on subalgebras of** $B[c]$. In this section we will establish that if Λ is any one of the nine algebras discussed in §§1 and 2, and if there exists a scalar homomorphism on Λ, then it is either χ or ρ.

Let θ denote the compact operators in $B[c]$. It is known that $A\in\theta\cap\Gamma$ if and only if $\sum_k |a_{nk}|$ converges uniformly with respect to n, and that $\Psi\supset\theta\cap\Gamma$ (see [2]).

3.1. LEMMA. *There is no scalar homomorphism on* $\theta\cap\Gamma$.

PROOF. Let $A\in\theta\cap\Gamma$ and for $j\in I^+$ let $C^j\in\theta\cap\Gamma$ be defined by $C^j_{ij}=a_{ij}$ for $i\in I^+$. Since $A\in\theta\cap\Gamma$, $A=\sum_{j=1}^{\infty} C^j$ where the convergence of the sum is in the usual norm topology of Γ. Since F is continuous,

$F(A)=\sum_{j=1}^{\infty} F(C^j)$. Let $j\in I^+$. Define R, $L\in\Gamma$ by $r_{j,j+1}=1$ and $l_{j+1,j}=1$. Then $R^2=L^2=0$ and $C^jRL=C^j$. Therefore, $F(C^j)=F(C^jRL)=0$.

3.2. THEOREM. *There is no scalar homomorphism on Ψ or on $M\cap\Psi$.*

PROOF. Let $B\in\Psi$. Define $B'\in\theta\cap\Gamma$ by $b'_{ij}=\lim_k b_{kj}$ for i, $j\in I^+$. Then letting $A=B-B'$ we get $A\in M\cap\Psi$ (because $\theta\cap\Gamma\subset\Psi$), and by Lemma 3.1 $F(A)=F(B)$. It, therefore, suffices to show that there is no scalar homomorphism on $M\cap\Psi$.

Let $A\in M\cap\Psi$ and assume $F(A)\neq 0$ for some scalar homomorphism. Write $A=A_1+A_2=A_3+A_4$, where A_1 (respectively A_2) is the result of replacing the even (respectively odd) rows of A by zeros, and where A_3 (respectively A_4) is the result of replacing the $4n-1$ and $4n$ (respectively $4n-3$ and $4n-2$) rows of A by zeros. Let P, Q, R, S be the elements of $M\cap\Psi$ defined as follows:

$$\begin{aligned}1 &= p_{2n,2n-1} = -p_{2n,2n} = q_{2n-1,2n} = -q_{2n-1,2n+1}\\ &= r_{4n-1,4n-3} = r_{4n,4n-2} = -r_{4n-1,4n-1} = -r_{4n,4n}\\ &= s_{4n-3,4n-1} = s_{4n-2,4n} = -s_{4n-3,4n+1} = -s_{4n-2,4n+2}.\end{aligned}$$

Then $QPA_1=A_1$, $PQA_2=A_2$, $SRA_3=A_3$, and $RSA_4=A_4$. Since $F(A)\neq 0$, either $F(A_1)\neq 0$ or $F(A_2)\neq 0$. In either case $F(P)\neq 0$ and $F(Q)\neq 0$. Similarly, $F(R)\neq 0$ and $F(S)\neq 0$. Now represent S by $S=S_1+S_2$, where S_1 (respectively S_2) is the result of replacing the $4n-2$ (respectively $4n-3$) rows of S by zeros. Since $F(S)\neq 0$, either $F(S_1)\neq 0$ or $F(S_2)\neq 0$. Either case yields a contradiction since $S_1P=0=S_2Q$. Hence, $F(A)\neq 0$ is impossible.

3.3. COROLLARY. *χ is the only scalar homomorphism on Γ and on $M\cap\Gamma$.*

3.4. THEOREM. *ρ is a scalar homomorphism on Ω.*

PROOF. This result follows from the fact that $\rho_\perp$ is an algebra (Theorem 1.5) and from [**3**, Lemma 3, p. 254].

3.5. THEOREM. *χ and ρ are the only possible scalar homomorphisms on any algebra in Ω which contains either Ψ or $M\cap\Gamma$.*

PROOF. Let Λ be such an algebra. As shown in §§1 and 2, Λ must be one of seven possible algebras. It follows that for any $T\in\Lambda$, $V(T)\in\Lambda$ and $B(T)\in\Lambda$. Therefore, for any scalar homomorphism F on Λ, $F(T)=F(V(T))+F(B(T))$. Since $\Lambda\subset\Omega$, $\{B(T): T\in\Lambda\}$ is a subalgebra of Γ containing either Ψ or $M\cap\Gamma$. Therefore, by Theorem 2.4, Corollary 3.3, and the fact that there are no subalgebras between Γ and Ψ, $F(B(T))=\chi(B(T))=\rho(T)$. Moreover, since $(V(T))^2 =(\lim_i \chi_i(T))V(T)$, either $F(V(T))=0$ or $F(V(T))=\lim_i \chi_i(T)$. Therefore, either $F(T)=\rho(T)$ or $F(T)=\lim_i \chi_i(T)+\rho(T)=\chi(T)$.

3.6. COROLLARY. *χ and ρ are the only scalar homomorphisms on $M\cap\Omega$ and on $M\cap\Omega_0$, and, in fact, $\chi=\rho$ on $M\cap\Omega_0$.*

3.7. LEMMA. *If $E\in\Lambda$ and if $F(E)\neq 0$ for some scalar homomorphisms F on Λ, then $F(T)=\lim Te$ for every $T\in\Lambda$.*

PROOF. Since $E^2=E$, $F(E)=1$. If $T\in\Lambda$, then $V(ET)=\chi(T)E$ and $B(ET)=B\in\Lambda$ where $b_{ij}=\lim Te^i$ for $i, j\in I^+$. Therefore, $F(T) =\chi(T)+F(B)$. But $F(B)=F(BE)=\sum_{i=1}^{\infty}\lim Te^i$, and so (by the definition of χ) $F(T)=\lim Te$.

3.8. THEOREM. *ρ is the only scalar homomorphism on Ω and there is no scalar homomorphism on $\rho_\perp$.*

PROOF. This is an immediate consequence of Theorem 3.5 and Lemma 3.7.

3.9. THEOREM. *χ is the only scalar homomorphism on M and there is no scalar homomorphism on $B[c]$.*

PROOF. It is clear that χ is a scalar homomorphism on M but not on $B[c]$. By noting that $E\in M$ and that $\chi(T)=\lim Te$ for $T\in M$ but not for all $T\in B[c]$, it suffices, by Lemma 3.7, to show that $F(E)\neq 0$ for any scalar homomorphism F on M or on $B[c]$.

Let I denote the identity operator. Then $F(I)=1$. Consider the following operators P, Q, R, S in M.

$$P_{2n,1} = Q_{2n+3,1} = R_{2n,1} = S_{2n+3,1} = -1;$$
$$P_{2n,2n} = Q_{2n+3,2n+3} = R_{2n,2n+1} = S_{2n+3,2n+2} = 1.$$

Then $P+Q=I-E$ and so if $F(E)=0$, either $F(P)\neq 0$ or $F(Q)\neq 0$. However, $RSP=P$, $RP=0$, $SRQ=Q$, and $SQ=0$, so that $F(P)=0$ and $F(Q)=0$. This contradiction establishes the theorem.

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STATE UNIVERSITY OF NEW YORK AT ALBANY