INVARIANT SUBSPACES WITH INVARIANT COMPLEMENTS

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1. **Introduction.** Let H_n denote the Hardy class of functions in H^2 of the unit disk Δ with values in the complex Hilbert space C_n. If $z\colon H_n \to H_n$ denotes the operator of multiplication by z and z^* is its operator conjugate on H_n, then one consequence of Theorem 1 of [3] is that the only projection[2] P on H_n which commutes with both z and z^* can be represented as a constant $n \times n$ matrix $P = [P_{ij}]$ which acts on H_n in the following way: if $u = \{u_j\}_{j=1}^n \in H_n$ then $Pu = \{\sum_j P_{ij} u_j\}_{i=1}^n$. An alternative interpretation is that the only orthogonal projection P on H_n which commutes with z is necessarily of the above form where $[P_{ij}]$ is an orthogonal projection on C_n. While orthogonal projection is the natural projection in Hilbert space it is still only one of many and we examine here the class of projections on H_n which commute with z but not necessarily with z^*. In H_1 $(=H^2)$ essentially nothing new happens. The consequences of Theorem 1 of [3] quoted above imply that if P is a projection on H_1 which commutes with z and z^* then $P=0$ or I. The same is true even if P merely commutes with z e.g. [4, Problem 116]. In H_2 things are different. If P denotes the matrix

$$\begin{pmatrix} z, & z \\ 1-z, & 1-z \end{pmatrix}$$

then P is a projection on H_2 which commutes with z, but is not a matrix of constants. The projection P decomposes H_2 into the direct sum $S \oplus T$ of two invariant subspaces

$$S = \{(u_1, u_2) = (zw, (1-z)w) \mid w \in H^2)\}$$

and

$$T = \{(u_1, u_2) = (w, -w) \mid w \in H^2\}.$$

The problem of determining the projections on H_n which commute with z is equivalent to the problem: Determine all translation invariant subspace S of H_n which have a translation invariant comple-

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[2] Unless stated otherwise, "projection" means "continuous projection" and "subspace" means "closed subspace."

ment in H_n, i.e. subspaces S such that $zS\subset S$ and $H_n=S\oplus T$ for some subspace T satisfying $zT\subset T$. Call such a subspace S *invariantly complemented*, or i.c. for brevity. The simplest nontrivial i.c. subspaces of H_n are of the form $S=AH_p$ where A is an $n\times p$ matrix of elements of H^∞ which by the addition of $n-p$ columns can be made into an isomorphism of H_n onto itself. In fact these are the only i.c. subspaces; see Theorem 2.1 below. The remaining results of this paper are extensions to matrices of elements from H^∞, of some elementary properties of matrices with complex entries. One of these, the corollary following Theorem 2.2′, is a natural extension of the Corona Theorem differing slightly from that of Fuhrmann [2].

The significant results needed here are the Corona Theorem [1], Lax's characterization of the invariant subspaces of H_n [5], [6], and the fact that any continuous linear operator $A: H_n\to H_n$ which commutes with z is representable as an $n\times n$ matrix of elements of H^∞ and A operates on H_n by matrix multiplication. Moreover if A is a topological linear isomorphism of H_n onto itself then $|\det A(z)|\geqq\epsilon>0$ for some ϵ and all $z\in\Delta$.

To conclude this introduction we note a property of i.c. subspaces which will not be used later, but which has some independent interest. If S is i.c. and $u\in H_n$ is such that $zu\in S$, then if P is a projection onto S commuting with z, $zu=Pzu=zPu$, i.e. $z(u-Pu)=0$ or $u=Pu\in S$. This extends immediately to polynomials, i.e. if $p(z)$ is a polynomial and $u\in H_n$ is such that $p(z)u\in S$, then $u\in S$, and taking weak limits is true even if $p(z)\in H^\infty$, since the polynomials are dense in the weak star topology on H^∞. This property is not shared by all invariant subspaces, e.g. if S is the invariant subspace z^2H^2 of H^2 and $p(z)=z$, then $z\in H^2$ and $z^2=p(z)\cdot z\in S$ but $z\notin S$. There is some evidence to suggest that this property is characteristic of i.c. subspaces. The following lemma is a sample of such evidence.

LEMMA 1.1. *Let $\phi_1,\cdots,\phi_n$ be elements of H^∞ such that $\sum|\phi_j|^2=1$ a.e. on $|z|=1$ and let $S\subset H_n$ be the invariant subspace of H_n, $\{(\phi_1u,\cdots,\phi_nu)\,|\,u\in H^2\}$. A necessary and sufficient condition that for every Blashke product $p(z)$, $u\in H_n$ and $pu\in S$ together imply $u\in S$, is that $\sum|\phi_j(z)|^2>0$ in Δ.*

The necessary and sufficient condition that S be i.c. is that $\sum|\phi_j(z)|^2\geqq\epsilon>0$ in Δ, (Theorem 2.2 below). The above lemma uses only Blashke products and not the whole of H^∞, but none the less is suggestive of the stronger result.

PROOF OF LEMMA 1.1. Note first that it is sufficient to consider only all p of the form $(z-a)/(1-\bar{a}z)$, for $|a|<1$. If $\sum|\phi_j(z)|^2>0$ and

$u=(u_1, \cdots, u_n)\in H_n$ is such that $u_i p=\phi_i v$ for all i and some $v\in H^2$ then $v(a)=0$ and so $p^{-1}v\in H^2$ whence $u\in S$ as required. Conversely, suppose that the ϕ_j have a common zero at $a\in\Delta$. Put $p(z)=(z-a)/(1-\bar{a}z)$ and $u=(\phi_1/p, \cdots, \phi_n/p)$. Then $u\in H_n$ and $pu\in S$ but $u\notin S$, completing the proof.

2. **Results.** In all that follows we adopt the convention that if S is a subspace of H_n and $z\in\Delta$, $S(z)$ is the subspace of C_n spanned by the numerical values of the elements of S at the point z; in particular $H_n(z)=C_n$. We also identify $H_p\times H_q$ with H_{p+q}.

LEMMA 2.1. *If there exists a continuous linear isomorphism T of H_n onto H_m which commutes with z then $m=n$.*

PROOF. Since $zT=Tz$, T can be represented as an $m\times n$ matrix of elements of H^∞. By assumption $C_m=T(z)C_n$ for any $z\in\Delta$ and so $n\geqq m$. Similarly T^{-1} as a map of H_m onto H_n being continuous by the closed graph theorem and also commuting with z, we have $n\leqq m$, and so $m=n$.

We now have our main theorem.

THEOREM 2.1. *If $S\subset H_n$ is an i.c. subspace then there exists a linear isomorphism A of H_n onto itself such that $zA=Az$ and $S=AP_pH_n$ where P_p denotes the projection of H_n onto H_p obtained by the coordinate projection $P_p\{u_1, \cdots, u_n\}=\{u_1, \cdots, u_p, 0, \cdots, 0\}$.*

PROOF. Since S is i.c. there does exist a continuous projection P of H_n onto S such that $zP=Pz$ and P may be realized as a matrix multiplication operator whose entries are elements of H^∞. $I-P$ also commutes with z and projects H_n onto a complementary subspace T. Both S and T are translation invariant and so by Lax's Theorem there exist $n\times p$ and $n\times q$ matrices A_S and A_T such that $A_S^*(z)A_S(z)=I$ and $A_T^*(z)A_T(z)=I$ for almost all z on $|z|=1$ and $S=A_SH_p$, $T=A_TH_q$. Here * denotes the conjugate. In particular A_S is invertible and $A_S^{-1}\colon S\to H_p$ is continuous and commutes with z. Similarly for T. Let $A\colon H_{p+q}=H_p\times H_q\to H_n$ be given by $A(u, v)=A_Su+A_Tv$, where $u\in H_p$, $v\in H_q$ and $B\colon H_n\to H_p\times H_q$ be given by $Bu=A_S^{-1}Pu+A_T^{-1}(I-P)u$, then $AB=I$, $BA=I$ and A is continuous and commutes with z. Thus by Lemma 2.1 $p+q=n$, and identifying H_{p+q} with H_n, we have $S=A_SH_p=AP_pH_n$ as asserted, and A clearly commutes with z.

Since A is invertible, so also is $A(z)$ and thus for all $z\in\Delta$, $\dim S(z)=p$. We shall refer to this constant integer p, improperly, as $\dim S$. As further consequences of Theorem 2.1 we now have that $S=H_n$, if

$S\subset H_n$, S is i.c., and dim $S=n$, cf. H^2 as mentioned in the introduction. Also the only irreducible i.c. subspaces of H_n have dim 1 and are isomorphic to H^2.

In order to give an analytic characterization of i.c. subspaces we shall need the estimate of the following elementary lemma.

LEMMA 2.2. *Let B be an invertible $n\times n$ matrix of complex numbers such that $|B_{ij}|\leqq\beta$. Suppose A and C are $n\times p$ ($p<n$) matrices such that $BA=C$. Then if $\{A^k\}$ ($\{C^k\}$) denotes the set of all $p\times p$ submatrices of A (C), we have*

$$(3.1)\qquad \sum_k |\det C^k| \geqq \det B\binom{n}{p}\beta^{p-n}[(n-p)!]^{-1}\sum_k |\det A^k|.$$

PROOF. Choosing any one of the matrices A^k, augment the matrix A by the addition, on the right, of $n-p$ columns each containing $n-1$ zeros and one one to obtain an $n\times n$ matrix $A^{(k)}$ such that $|\det A^{(k)}| = |\det A^k|$. Then C is the matrix of the first p columns of $BA^{(k)}$. Expanding det $BA^{(k)}$ by elements of the last row, then expanding each of the $(n-1)$th order determinants so obtained by elements of their last rows and repeating the process as often as necessary, we obtain

$$|\det B|\,|\det A^k| \leqq \sum_j |\det C^j|\,\beta^{n-p}(n-p)!$$

On summing this inequality over all submatrices A^k and rewriting, we obtain the inequality (3.1).

The estimate of this lemma is certainly very poor but it is sufficient for what follows.

THEOREM 2.2. *Let $S=AH_p$ (where $A^*A=I$ on $|z|=1$) be a translation invariant subspace of H_n of* dim p. *Then S is i.c. if and only if there exists an $\epsilon>0$ such that*

$$(3.2)\qquad \sum |\det A^k(z)| \geqq \epsilon \qquad \textit{for all } z\in\Delta,$$

where $\{A^j(z)\}$ is the set of all $p\times p$ submatrices of the $n\times p$ matrix $A(z)$.

PROOF. Suppose that S is i.c. By Theorem 2.1 there exists a matrix $B(z)$ of elements of H^∞ such that, for all $z\in\Delta$, $|\det B(z)|\geqq\alpha>0$, $|B_{ij}(z)|\leqq\beta$ and $(B(z)A(z))_{ij}=\delta_{ij}$ for $i\leqq p$, and zero otherwise. By Lemma 2.2 the inequality (3.2) now holds for some ϵ.

Conversely suppose that (3.2) is valid. Assume first that $p=1$, i.e. $A(z)=\{A_j(z)\}$, $j=1,\cdots,n$, and $\sum|A_j(z)|\geqq\epsilon$ in Δ. By the Corona Theorem [1] there exist functions $\theta_j(z)\in H^\infty$ such that $\sum_j\theta_jA_j=1$ in

Δ. Let $P: H_n \to H_n$ be the matrix operator given by $P_{ij} = A_i\theta_j$, $i, j = 1, \cdots, n$, then P is a projection onto S which commutes with z. Thus S is i.c. as required.

Now assume the theorem true for all S with dim $S < p$ $(p > 1)$. If (3.2) holds it is easily checked that for some $\epsilon > 0$, $\sum_j |A_{jp}(z)| \geqq \epsilon$ and $\sum |\det B^k(z)| \geqq \epsilon$ in Δ, where B is the $n \times (p-1)$ matrix obtained from A by deleting the last column. By assumption then, both $S' = \{\{A_{jp}u\} | u \in H^2\}$ and $S'' = BH_{p-1}$ are i.c. and $S = S' + S''$. Let D^{-1} be the isomorphism of Theorem 2.1 such that $S'' = D^{-1}P_{p-1}H_n$ then with the proper identifications, $(DB)_{ij} = \delta_{ij}$ if $i \leqq p-1$ and zero otherwise while $(DA)_{ij} = \delta_{ij}$ if $i \leqq p-1$, and zero if $i > p-1$ & $j < p-1$, or $j > p$. But applying Lemma 2.2 we have that

$$\sum_k |\det(DA)^k| \geqq \epsilon'$$

for some ϵ'. However now

$$\sum_k |\det(DA)^k| = \sum_{j=p}^{n} |(DA)_{jp}|.$$

Thus $S_1 = \{\{(DA)_{jp}u\}_{j=p}^{n} | u \in H^2\}$ is an i.c. subspace of H_{n-p+1} of dim 1. Also in this coordinate system $S'' = H_{p-1}$. But now $S = S' + S'' = S_1 \oplus H_{p-1}$ and since $H_{n-p+1} \cap H_{p-1} = 0$, and S_1 is an i.c. subspace of H_{n-p+1}, S is itself i.c. Since the property of being i.c. is rather trivially preserved under linear isomorphisms of H_n which commute with z, S in the original coordinate system is i.c. as required.

The last paragraph of the proof of Theorem 2.2 yields with a little modification a necessary and sufficient condition that the sum of two i.c. subspaces be i.c. We omit the proof but state the result.

THEOREM 2.2′. *If $S = AH_p$ and $S' = A'H_q$ are i.c. subspaces of* dim p *and q respectively, then a necessary and sufficient condition that $S + S'$ be i.c. of* dim $p+q$ *is that*

$$\sum |\det C^k| \geqq \epsilon > 0 \qquad \text{in } \Delta$$

where $\{C^k\}$ is the set of $(p+q) \times (p+q)$ submatrices of the $n \times (p+q)$ matrix C whose columns are those of A and those of A'.

As a statement about matrices of elements from H^∞, Theorem 2.2 has an interesting interpretation. Observing that the fact $A^*A = I$ was not used in the proof but appeared merely because we chose to use the Lax representation for S, we have on combining Theorems 2.1 and 2.2, COROLLARY. *If A is an $n\times p$ matrix of elements from H^∞ then there exists a nonsingular matrix B of elements from H^∞ such that $(BA)_{ij}=\delta_{ij}$ if $i\leqq p$ and zero otherwise if and only if $\sum|\det A^k|\geqq\epsilon>0$ in Δ.*

A particular case of this Corollary (essentially the case $n=rp$, r an integer, but without showing B may be chosen nonsingular) has been given by Fuhrmann [**2**, Theorem 3.1].

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