# PRODUCTS OF TWO ONE-PARAMETER SUBGROUPS

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Let **R** denote the field of real numbers.

THEOREM. *Let $G$ be a Lie group with Lie algebra $L$. Let $X$ and $Y$ be linearly independent elements of $L$. If the set* $\exp \mathbf{R}X\cdot\exp \mathbf{R}Y$ *contains a one-parameter subgroup* $\exp \mathbf{R}Z$ *such that* $\exp \mathbf{R}X \not\supset \exp \mathbf{R}Z$ *and* $\exp \mathbf{R}Y \not\supset \exp \mathbf{R}Z$, *then* $\{X, Y\}$ *forms a basis of a two-dimensional subalgebra of $L$.*

PROOF. We can find a positive number $\epsilon$ and analytic functions $\phi(t)=a_1t+a_2t^2+\cdots$ and $\psi(t)=b_1t+b_2t^2+\cdots$, defined in the interval $(-\epsilon, \epsilon)$, such that we have

$$\exp tZ = \exp \phi(t)X\cdot\exp \psi(t)Y, \qquad -\epsilon < t < \epsilon.$$

On the other hand, for real numbers $\phi$ and $\psi$, sufficiently close to 0, we have

$$\exp \phi X\cdot\exp\psi Y = \exp\{(\phi X+\psi Y)+\tfrac{1}{2}[\phi X, \psi Y]+\cdots\}.$$

Hence for $t\in\mathbf{R}$, with $|t|$ small enough, we have

$$\begin{aligned} tZ &= \phi(t)X+\psi(t)Y+\tfrac{1}{2}[\phi(t)X, \psi(t)Y]+\cdots \\ &= t(a_1X+b_1Y)+t^2(a_2X+b_2Y+\tfrac{1}{2}a_1b_1[X, Y])+\cdots, \end{aligned}$$

and it follows that

$$Z = a_1X+b_1Y, \qquad a_2X+b_2Y+\tfrac{1}{2}a_1b_1[X, Y]=0.$$

Because $a_1\neq 0$ and $b_1\neq 0$, we have that

$$[X, Y] = -\frac{2}{a_1b_1}(a_2X+b_2Y).$$

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