RIESZ MATRICES THAT ARE ALSO HAUSDORFF MATRICES

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If $A = (a_0, a_1, a_2, \cdots)$ is an infinite sequence, $a_0 > 0$, $a_p \geqq 0$ for $p \geqq 1$ and S is the sequence of partial sums of A then by the Riesz matrix for A we shall mean the infinite triangular matrix $[A]$ such that $[A]_{np} = a_p/S_n$, $0 \leqq p \leqq n$ and $[A]_{np} = 0$ otherwise.

If $D = (d_0, d_1, d_2, \cdots)$ is an infinite number sequence, $\Delta^0 d_p = d_p$ and, for each n, $\Delta^{n+1} d_p = \Delta^n d_p - \Delta^n d_{p+1}$, then the Hausdorff matrix for the sequence D is the infinite triangular matrix $[H(D)]$ such that $[H(D)]_{np} = C_{np}\Delta^{n-p} d_p$, $0 \leqq p \leqq n$, and $[H(D)]_{np} = 0$ otherwise.

In considering the Riesz matrix for the sequence A we may, without loss of generality, consider only sequences with $a_0 = 1$. It is well known that the Riesz matrix for the sequence A is regular only in case the sequence of partial sums of A diverges, and that the Hausdorff matrix for the sequence D is regular only in case there exists a function, α, of bounded variation on $[0, 1]$, such that $\alpha(0+) = \alpha(0)$, $\alpha(1) - \alpha(0) = 1$ and for each nonnegative integer p, $d_p = \int_0^1 t^p d\alpha(t)$.

It was noticed by Garabedian and Wall [1, pp. 198–199] that a certain class of hypergeometric summability matrices were also Riesz matrices. The following theorem establishes necessary and sufficient conditions that a Riesz matrix for the sequence A also be a regular Hausdorff matrix.

THEOREM. *If A is a sequence such that $a_0 = 1$ and for each $p \geqq 1$, $a_p \geqq 0$ and $[A]$ is the Riesz matrix for A then the following two statements are equivalent*:

(1) *There is a positive number x such that $A = (1, x, x(x+1)/2, \cdots, x(x+1) \cdots (x+p-1)/p!, \cdots)$.*

(2) *The matrix $[A]$ is a regular Hausdorff matrix.*

Suppose (1) is true. The sequence of partial sums of A is as follows; $S_0 = 1$ and if $p \geqq 1$ then $S_p = (1+x)(2+x) \cdots (p+x)/p!$. A short computation will show that if each of n and p is a nonnegative integer, $p \leqq n$, then $[A]_{np} = C_{np}(x(n-p)!/(x+p)(x+p+1) \cdots (x+n))$. In particular $[A]_{nn} = x/(x+n)$. An induction argument shows that if each of n and p is a nonnegative integer, $p \leqq n$, then $\Delta^n[A]_{pp} = (n!)x/(x+p)(x+p+1) \cdots (x+p+n)$. In particular

$$C_{np}\Delta^{n-p}[A]_{pp} = C_{np}((n-p)!x)/(x+p) \cdots (x+n) = [A]_{np}.$$

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Therefore $[A]$ is a Hausdorff matrix. The function $\alpha(t)=t^x$ on $(0, 1]$ with $\alpha(0)=0$ has the property that $\int_0^1 t^n d\alpha(t)=x/(x+n)$, and hence $[A]$ is a regular matrix.

Suppose (2) is true. If $[A]$ is a Riesz matrix that is also a Hausdorff matrix then $[A]_{(n+1)n}=(n+1)([A]_{nn}-[A]_{(n+1)(n+1)})$ for each n. Moreover $[A]_{(n+1)n}=a_n/S_{(n+1)}$ and the following recursion formula is obtained: $a_n/S_{n+1}=(n+1)((a_n/S_n)-(a_{(n+1)}/S_{(n+1)}))$. This may be solved for a_{n+1} and we then obtain the formula $a_{n+1}=(n/n+1)a_n(S_n/S_{n-1})$, which shows that the sequence A is completely determined by the term a_1.

If $a_1=0$ then $A=(1, 0, 0, 0, \cdots)$ and $[A]$ is not regular. It is, however, a semiregular Hausdorff matrix. Hence $a_1>0$. An induction argument shows that

$$A = \frac{(1, a_1, a_1(1+a_1)}{2}, \cdots, \frac{a_1(1+a_1)(2+a_1)\cdots(n-1+a_1)}{n!}, \cdots).$$

This concludes the proof of the theorem.

Hausdorff has shown the summability method defined by a Riesz matrix of the type designated in the theorem is equivalent to $(C, 1)$ summability [1].

Reference

1. H. L. Garabedian and H. S. Wall, *Hausdorff methods of summation and continued fractions*, Trans. Amer. Math. Soc. **48** (1940), 185–207.

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