PRODUCTS OF k'-SPACES

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We call a topological space X a k'-space if $A \subset X$, $x \in \bar{A}$ implies the existence of a compact subset K of X such that $x \in \mathrm{Cl}(A \cap K)$ (where Cl stands for closure) [1]. A characterization of k'-spaces is given in [8]. E. A. Michael has shown that a Hausdorff space is locally compact if its product with every k-space is a k-space [6]. We obtain an analogous result for k'-spaces which implies the existence of k-spaces which are not k'-spaces. It is stated in [1] that such spaces exist as opposed to remarks in [4] and [7]. We show that a T_1 space X is discrete if $X \times Y$ is a k'-space for every k'-space Y. Thus a nontrivial product theorem for k'-spaces must involve additional conditions on both factors, in contrast to Cohen's Theorem [3] ($X \times Y$ is a k-space for X locally compact and Y a k-space—see for instance [2]). We do show that $X \times Y$ is a k'-space if both X and Y are T_1, k'-spaces and $X \times Y$ has a nested neighborhood base at each of its points. Also the product of two T_1 spaces with a nested neighborhood base at each point is a k-space if one of the spaces is a k'-space and the other is a k-space.

THEOREM 1. *If X is a nondiscrete T_1 space, then there is a k'-space Y such that $X \times Y$ is not a k'-space.*

PROOF. Let $\{x_\alpha : \alpha \in D\}$ be a net converging to x such that $x_\alpha \neq x$, $\alpha \in D$. Let $Y_1 = \{(\alpha, n) : \alpha \in D$ and $n = 1, 2, 3, \cdots\}$ and let $Y = Y_1 \cup \{z\}$. The topology on Y is as follows: Y_1 is discrete and the open sets containing z contain all but a finite number of elements of each set $A = \{(\alpha, n) : n = 1, 2, 3, \cdots\}$ for $\alpha \in D$. It is easy to see that Y is a k'-space since each compact subset intersects only finitely many of the sets A. On the other hand (x, z) is a limit point of the set $C = \{(x_\alpha, (\alpha, n)) : \alpha \in D, n = 1, 2, 3, \cdots\}$ but clearly not a limit point of $C \cap K$ for any compact set K. Thus $X \times Y$ is not a k'-space.

REMARK. If X is a nondiscrete locally compact T_1 space, then there is a k'-space Y (as in the proof of the theorem) such that $X \times Y$ is a k-space which is not a k'-space. The space Y is paracompact. As a matter of fact every open cover has a discrete open refinement. Also Y can be slightly modified to make it a CW-complex.

Before proving Theorem 2 we need a lemma on product spaces with nested neighborhood bases. From this point on we assume that X and Y are T_1 spaces.

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LEMMA. *If $X\times Y$ has a nested neighborhood base at $(x, y)\in\bar{A}-A$ and there are neighborhoods U of x and V of y such that $\{x\}\times V\cap A=\emptyset$ and $U\times\{y\}\cap A=\emptyset$, then there is a net $\{(x_\alpha, y_\alpha):\alpha\in D\}$ in A which converges to (x, y) and, for each $\alpha_0\in D$, there are neighborhoods R of x and S of y such that $x_\alpha\notin R$, $y_\alpha\notin S$ for $\alpha<\alpha_0$.*

PROOF. We can choose a nested neighborhood base at (x, y) of the form $\{U_\alpha\times V_\alpha:\alpha\in D\}$ where D is directed as follows: $\alpha<\beta$ iff $U_\alpha\times V_\alpha\supset U_\beta\times V_\beta$. We well order the set $\mathfrak{B}=\{U_\alpha\times V_\alpha:\alpha\in D\}$ by $\prec$ and for each $\alpha\in D$ choose the first element $U_\beta\times V_\beta$ of $\mathfrak{B}$ such that $(U_\alpha-U_\beta\times V_\alpha-V_\beta)\cap A\neq\emptyset$. If for some fixed α no such choice is possible, then $A\cap(\bigcup\{U_\alpha-U_\beta\times V_\alpha-V_\beta:\beta\in D\})=\emptyset$. Since $\mathfrak{B}$ is nested, $\bigcup\{U_\alpha-U_\beta\times V_\alpha-V_\beta\}=U_\alpha-\{x\}\times V_\alpha-\{y\}$. We can assume that $U_\alpha\times V_\alpha\subset U\times V$ where U and V are the neighborhoods given in the hypothesis. Thus, $\emptyset=A\cap(U_\alpha-\{x\}\times V_\alpha-\{y\}\cup U_\alpha\times\{y\}\cup\{x\}\times V_\alpha)\supset A\cap(U_\alpha\times V_\alpha-\{(x, y)\})$. This contradicts the fact that $(x, y)\in\bar{A}-A$. Now, for each $\alpha\in D$, we take $(x_\alpha, y_\alpha)\in(U_\alpha-U_\beta\times V_\alpha-V_\beta)\cap A$ where $U_\beta\times V_\beta$ is chosen as indicated above. Let $\alpha_0\in D$ and let $R=U_{\beta_0}$, $S=V_{\beta_0}$ again choosing β_0 such that $U_{\beta_0}\times V_{\beta_0}$ is the first element of $\mathfrak{B}$ satisfying $(U_{\alpha_0}-U_{\beta_0}\times V_{\alpha_0}-V_{\beta_0})\cap A\neq\emptyset$. We complete the proof by showing that $x_\alpha\notin U_{\beta_0}$ and $y_\alpha\notin V_{\beta_0}$ for $\alpha<\alpha_0$. Suppose there is $\alpha<\alpha_0$ such that $x_\alpha\in U_{\beta_0}$ or $y_\alpha\in V_{\beta_0}$. Since $U_{\beta_0}\times V_{\beta_0}\subset U_{\alpha_0}\times V_{\alpha_0}\subset U_\alpha\times V_\alpha$, $(U_\alpha-U_{\beta_0}\times V_\alpha-V_{\beta_0})\cap A\neq\emptyset$. It follows that $U_\beta\times V_\beta\prec U_{\beta_0}\times V_{\beta_0}$. Since $x_\alpha\notin U_\beta$ and $y_\alpha\notin V_\beta$, we have $U_\beta\times V_\beta\subset U_{\alpha_0}\times V_{\alpha_0}$. This implies that $U_{\beta_0}\times V_{\beta_0}\prec U_\beta\times V_\beta$ which is a contradiction. This completes the proof.

By a routine use of the definition of subnet [5] we can establish the fact that any net in $\{x_\alpha\}$ (resp. $\{y_\alpha\}$) which converges to x (resp. y) is a subnet of $\{x_\alpha\}$ (resp. $\{y_\alpha\}$). We use this fact in the proof of Theorem 2. We also use the following characterization of k-spaces established in [8]: A topological space X is a k-space iff, for each subset A and $x\in\bar{A}$, there is a closed k-subspace C such that $x\in\mathrm{Cl}(A\cap C)$.

THEOREM 2. *If X is a k'-space and Y is a k'-space (k-space) and $X\times Y$ has a nested neighborhood base at each point, then $X\times Y$ is a k'-space (k-space).*

PROOF. Let A be a subset of $X\times Y$ and let $x\in\bar{A}-A$. If the neighborhoods U and V of the lemma do not exist, then our conclusion follows routinely. Thus there is a net $\{(x_\alpha, y_\alpha)\}$ in A converging to (x, y) and satisfying the conclusion of the lemma. Since X is a

k'-space, there is a compact subset K of X such that $x \in \text{Cl}(\{x_\alpha\} \cap K)$. Thus there is a net $\{x_\gamma\}$ in $\{x_\alpha\} \cap K$ which converges to x. By the note which follows the proof of the lemma $\{x_\gamma\}$ is a subnet of $\{x_\alpha\}$ and $\{y_\gamma\}$ converges to y, being a subnet of $\{y_\alpha\}$. Since Y is a k-space, there is a closed k-subspace C of Y (in case Y is a k'-space C can be chosen compact) such that $y \in \text{Cl}(\{y_\gamma\} \cap C)$. Finally we obtain a subnet of $\{(x_\alpha, y_\alpha)\}$ in $K \times C \cap A$. Thus $(x, y) \in \text{Cl}(K \times C \cap A)$. If C is a k-space, then $K \times C$ is a k-space [2] and if C is compact $K \times C$ is also. Thus, in case Y is a k-space, $X \times Y$ is a k-space and a k'-space when Y is.

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