IDENTIFYING PERTURBATIONS WHICH PRESERVE ASYMPTOTIC STABILITY[1]

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1. If the zero solution is uniform-asymptotically stable for the vector ordinary differential equation

$$\text{(E)} \qquad x' = f(t, x),$$

then it is also uniform-asymptotically stable for the perturbed equation

$$\text{(P)} \qquad y' = f(t, y) + g(t, y)$$

if f satisfies a uniform Lipschitz condition and if g is "sufficiently small." Such sufficiently small g we will call *permissible*. This result is known and the proof is essentially the same as the proof that a Lipschitz, uniform-asymptotically stable system is totally stable [**1**, p. 276]; namely, a positive definite, decrescent Lyapunov function V exists for (E) satisfying $\dot{V}_E(t, x) \leqq -c(|x|)$ and $|\text{grad } V(t, x)| \leqq b$. Therefore

$$V_P(t, x) = \dot{V}_E(t, x) + \langle \text{grad } V(t, x), g(t, x) \rangle \leqq -\tfrac{1}{2}c(|x|)$$

if $|g(t, x)| \leqq c(|x|)/2b$. Thus an estimate on the size of permissible perturbations g is provided in terms of a Lyapunov function associated with (E). If f has further special properties, so might V. In this way Hahn [**1**, p. 282] proved the following:

Let $f_x(t, x)$ be continuous and bounded for $t \geqq 0$ and $|x| \leqq 1$. Suppose that for all real c and some $k \geqq 1$, $f(t, cx) = c^k f(t, x)$. Then $g(t, x) = o(|x|^k)$ is permissible.

The purpose of this paper is to establish estimates on the size of permissible g in terms of the rate of approach to zero of the solutions of (E). Using these estimates we can prove Hahn's theorem without assuming that f is differentiable.

2. Let R^n denote Euclidean n-space. Let $\langle x, y \rangle$ denote the inner product of x and y in R^n, i.e., $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$. Let $|x|$

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$=\langle x, x\rangle^{1/2}$. Consider (E) and (P) where f and g map $[0, \infty)\times R^n$ continuously into R^n. Assume $f(t, 0)=g(t, 0)=0$. Thus for each $t_0\geqq 0$ and each $x_0\in R^n$, there is at least one solution $x(t; t_0, x_0)$ of (E) and at least one solution $y(t; t_0, x_0)$ of (P) through (t_0, x_0) which are defined for t in a neighborhood of t_0. (We do not assume that the solutions of (E) or (P) are uniquely determined by (t_0, x_0).)

DEFINITION 2.1. The zero solution is *uniform-asymptotically stable* (UAS) for (E) if (i) for every $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that $|x(t; t_0, x_0)|<\epsilon$ for all $|x_0|<\delta$ and $t\geqq t_0\geqq 0$, and if (ii) there exists $\delta_0>0$ and for every $\eta>0$ there exists $T=T(\eta)\geqq 0$ such that $|x(t; t_0, x_0)|<\eta$ for $|x_0|<\delta_0$, $t_0\geqq 0$, and $t\geqq t_0+T$.

If the solutions of (E) are not uniquely determined by (t_0, x_0), then the zero solution is UAS provided that (i) and (ii) above hold for all the solutions through (t_0, x_0).

Following Hahn [**1**, p. 7] we say that a real-valued function $\phi(\cdot)$ *belongs to class* K if, for some $r_1>0$, $\phi(\cdot)$ is continuous and strictly increasing on $[0, r_1]$ and $\phi(0)=0$.

3. We begin with a lemma which characterizes uniform-asymptotic stability in terms of certain auxiliary functions. These functions appear to be more useful for perturbation problems than those of Hahn [**1**, p. 8]; however, Hahn's functions seem more useful for converse theorems on Lyapunov functions [**1**, Chapter 6].

LEMMA 3.1. *The zero solution of* (E) *is UAS if and only if there exist functions* $\alpha(\cdot)$ *and* $\beta(\cdot)$ *in* K *and a positive function* $\tau(\cdot)$ *such that*

$$\alpha(\delta) < \delta \leqq \beta(\delta) \qquad \text{for } 0 < \delta < \delta_0, \tag{3.1}$$

and for all $|x_0| \leqq\delta<\delta_0$, $t_0\geqq 0$, *and* $t_0\leqq t\leqq t_0+\tau(\delta)$,

$$|x(t; t_0, x_0)| \leqq \beta(\delta) \quad \text{and} \quad |x(t_0+\tau(\delta); t_0, x_0)| \leqq \alpha(\delta). \tag{3.2}$$

PROOF. If the zero solution is UAS, then $\beta(\cdot)$ exists by [**1**, p. 173]. Choose $\alpha(\delta)=\frac{1}{2}\delta$. Now we can take $\tau(\delta)=T(\frac{1}{2}\delta)$ by Definition 2.1.

Conversely, suppose (3.1) and (3.2) hold. Let $\epsilon>0$. Let $\delta=\delta(\epsilon)$ exist so that $0<\delta<\delta_0$ and $\beta(\delta)<\epsilon$. Let $|x_0|\leqq\delta$ and $t_0\geqq 0$. Then

$$|x(t_0+\tau(\delta): t_0, x_0)| \leqq \alpha(\delta) < \delta.$$

Therefore (3.2) holds with t_0 replaced by $t_0+\tau(\delta)$ and x_0 replaced by $x(t_0+\tau(\delta); t_0, x_0)$. Thus

$$|x(t_0+2\tau(\delta); t_0, x_0)| \leqq \alpha(\delta) < \delta;$$

hence (3.2) holds with t_0 replaced by $t_0+2\tau(\delta)$. By induction

$|x(t;\ t_0,\ x_0)| \leqq \beta(\delta) < \epsilon$ for all $t \geqq t_0$. Thus (i) of Definition 2.1 holds.

Now choose $\delta_n = \alpha(\delta_{n-1})$ and $t_n = \tau(\delta_{n-1}) + t_{n-1}$ for each $n = 1, 2, \cdots$. Since $\{\delta_n\}$ is a decreasing sequence of positive numbers, there exists $\vartheta \geqq 0$ such that $\delta_n \to \vartheta$ as $n \to \infty$. But $\delta_n - \alpha(\delta_n) \to \vartheta - \alpha(\vartheta)$ and $\delta_n - \alpha(\delta_n) = \delta_n - \delta_{n+1} \to 0$. Hence $\alpha(\vartheta) = \vartheta$. By (3.1), $\vartheta = 0$. Let $\eta > 0$. Choose $N = N(\eta)$ so large that $\delta_N < \delta(\eta)$, where $\delta(\eta)$ comes from (i) of Definition 2.1. Consider any solution $\bar{x}(\cdot;\ t_0,\ x_0)$ through $(t_0,\ x_0)$. Then

$$|\bar{x}(t_1;\ t_0,\ x_0)| = |\bar{x}(t_0 + \tau(\delta_0);\ t_0,\ x_0)| \leqq \alpha(\delta_0) = \delta_1.$$

Therefore, for some solution $x(\cdot;\ t_1,\ \bar{x}(t_1;\ t_0,\ x_0))$,

$$|\bar{x}(t_2;\ t_0,\ x_0)| = |x(t_2;\ t_1,\ \bar{x}(t_1;\ t_0,\ x_0))| \leqq \alpha(\delta_1) = \delta_2.$$

By repeating this argument, we have that

$$|\bar{x}(t_N;\ t_0,\ x_0)| = |x(t_N;\ t_{N-1},\ \bar{x}(t_{N-1};\ t_0,\ x_0))| \leqq \alpha(\delta_{N-1}) = \delta_N.$$

Since $\delta_N < \delta(\eta)$, it follows that

$$|\bar{x}(t;\ t_0,\ x_0)| = |x(t;\ t_N,\ \bar{x}(t_N;\ t_0,\ x_0))| < \eta$$

for all $t \geqq t_N = t_0 + T(\eta)$ for some solution $x(\cdot;\ t_N,\ \bar{x}(t_N;\ t_0,\ x_0))$, where

$$T(\eta) = \tau[\alpha^{(N-1)}(\delta_0)] + \tau[\alpha^{(N-2)}(\delta_0)] + \cdots + \tau[\delta_0].$$

Thus (ii) of Definition 2.1 holds. Hence the zero solution is UAS and Lemma 3.1 is proved.

We now restrict f somewhat and prove a result concerning the distance of a solution of (P) from one of (E). A similar result appears also in [2, Lemma 5.1].

Lemma 3.2. *Suppose that for some γ in the class K, some $L > 0$, some $r > 0$, and all $|x| \leqq r$, $|y| \leqq r$, and $t \geqq 0$, we have*

$$\langle x - y, f(t,\ x) - f(t,\ y)\rangle \leqq L|x - y|^2, \tag{3.3}$$

$$|g(t,\ x)| \leqq \gamma(|x|). \tag{3.4}$$

Let $u > 0$, $t_0 \geqq 0$, and let $|x(t;\ t_0,\ x_0)| \leqq r$ and $|y(t;\ t_0,\ x_0)| \leqq r$ for $t_0 \leqq t \leqq t_0 + u$. Then for all $t_0 \leqq t \leqq t_0 + u$,

$$|x(t;\ t_0,\ x_0) - y(t;\ t_0,\ x_0)| \leqq 2\gamma(r)ue^{2Lu}.$$

Remark. If f satisfies a uniform Lipschitz condition, i.e., $|f(t,\ x) - f(t,\ y)| \leqq L|x - y|$ for all $t \geqq 0$, $|x| \leqq r$, and $|y| \leqq r$, then f satisfies (3.3). Of course the converse is false, e.g. $f(t,\ x) = -tx^3$. If f satisfies (3.3), then solutions of (E) are uniquely determined by $(t_0,\ x_0)$ for $t > t_0$ but not necessarily for $t < t_0$. Even in this case solutions of (P) need not be uniquely determined for $t > t_0$.

PROOF. Let $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; t_0, x_0)$. Define $\lambda = \sup |x(t) - y(t)|$ for $t_0 \leqq t \leqq t_0 + u$. Then

$$\langle x'(t) - y'(t), x(t) - y(t)\rangle = \langle x(t) - y(t), f(t, x(t)) - f(t, y(t))\rangle - \langle x(t) - y(t), g(t, y(t))\rangle;$$

hence

$$| x(t) - y(t) |^2 \leqq 2\lambda\gamma(r)u + \int_{t_0}^{t} 2L \, | x(s) - y(s) |^2 ds.$$

By Gronwall's inequality

$$| x(t) - y(t) |^2 \leqq 2\lambda\gamma(r)ue^{2Lu}$$

for all $t_0 \leqq t \leqq t_0 + u$. Therefore $\lambda^2 \leqq 2\lambda\gamma(r)ue^{2Lu}$ from which the result follows.

4. Our main result says that if f satisfies (3.3) and if the zero solution is UAS for (E) with corresponding $\alpha_E(\cdot)$, $\beta_E(\cdot)$, and $\tau(\cdot)$, then by choosing appropriate larger $\alpha_P(\cdot)$ and $\beta_P(\cdot)$, there will be room enough to perturb (E) by certain functions g and still have that the zero solution is UAS, but with corresponding $\alpha_P(\cdot)$, $\beta_P(\cdot)$ and the same $\tau(\cdot)$.

THEOREM 4.1. *Let f satisfy* (3.3). *Let the zero solution of* (E) *be UAS with corresponding* $\alpha_E(\cdot)$, $\beta_E(\cdot)$, and $\tau(\cdot)$. *Suppose there exist* $\alpha_P(\cdot)$, $\beta_P(\cdot)$, *and* $\gamma(\cdot)$ *in the class K such that for some $r > 0$ and all $0 < \delta \leqq r$, we have*

$$\alpha_E(\delta) < \alpha_P(\delta) < \delta \leqq \beta_E(\delta) < \beta_P(\delta), \tag{4.1}$$

$$\gamma(\beta_P(\delta)) < [2\tau(\delta)e^{2L\tau(\delta)}]^{-1} \min\{\beta_P(\delta) - \beta_E(\delta), \alpha_P(\delta) - \alpha_E(\delta)\}. \tag{4.2}$$

Then if $|g(t, x)| \leqq \gamma(|x|)$ for $t \geqq 0$ and $|x| \leqq r$, the zero solution of (P) *is UAS with corresponding* $\alpha_P(\cdot)$, $\beta_P(\cdot)$, *and* $\tau(\cdot)$.

REMARKS. Note that the right-hand side of (4.2) is positive because of (4.1). Since $\beta_P(\cdot)$ is strictly increasing, $\gamma(\cdot)$ is well-defined by (4.2). Observe the structure of (4.2): the bound $\gamma(\cdot)$ for g depends on the choices of $\beta_P(\cdot)$ and $\alpha_P(\cdot)$. To make $\gamma(\cdot)$ larger, one must take $\alpha_P(\cdot)$ closer to the identity function and thus obtain a slower approach to zero of the solutions of (P). Actually, since $\alpha_E(\cdot)$, $\beta_E(\cdot)$, and $\tau(\cdot)$ are not uniquely determined, some manipulating of these might result in better estimates for $\gamma(\cdot)$. This can be complicated because, for example, decreasing $\alpha_E(\cdot)$ would seem to force the increasing of $\tau(\cdot)$ which might make the right-hand side of (4.2) even smaller. The difficult but important problem of juggling all these scalar functions in order to obtain the best estimate of $\gamma(\cdot)$ from (4.2) has not been solved as yet. In some cases, it seems helpful to choose $\alpha_E(\cdot)$ in such a way that $\tau(\cdot)$ is constant (see the proof of Theorem 5.1). Example 8.2 of [2] shows that Theorem 4.1 need not hold if f does not satisfy (3.3).

PROOF. Let $|x_0| < \delta \leqq r$ and $t_0 \geqq 0$. Let $x(\cdot)$ and $y(\cdot)$ be solutions of (E) and (P), respectively, through (t_0, x_0). For as long as $|y(t)| \leqq \beta_P(\delta)$ on the interval $t_0 \leqq t \leqq t_0 + \tau(\delta)$, we have

$$\begin{aligned} |y(t)| &\leqq |x(t)| + |y(t) - x(t)| \\ &\leqq \beta_E(\delta) + 2\gamma(\beta_P(\delta))\tau(\delta)e^{2L\tau(\delta)} < \beta_P(\delta). \end{aligned}$$

Thus $|y(t)| < \beta_P(\delta)$ for $t_0 \leqq t \leqq t_0 + \tau(\delta)$. Also

$$\begin{aligned} |y(t_0 + \tau(\delta))| &\leqq |x(t_0 + \tau(\delta))| + |y(t_0 + \tau(\delta)) - x(t_0 + \tau(\delta))| \\ &\leqq \alpha_E(\delta) + 2\gamma(\beta_P(\delta))\tau(\delta)e^{2L\tau(\delta)} < \alpha_P(\delta). \end{aligned}$$

By Lemma 3.1, the zero solution is UAS for (P). This completes the proof.

5. We now apply Theorem 4.1 to obtain

THEOREM 5.1 *Let f satisfy* (3.3) *and for all real c and some $k \geqq 1$ let*

$$f(t, cx) = c^k f(t, x). \tag{5.1}$$

Let the zero solution of (E) *be UAS. Then if $g(t, x) = o(|x|^k)$, the zero solution of* (P) *is UAS.*

REMARK. Hahn [1, p. 282] proved this result by using Lyapunov functions and under the additional assumption that f has continuous first partial derivatives with respect to x which are uniformly bounded with respect to t. Note that if f is linear in x, then f satisfies (5.1) with $k=1$.

PROOF. First, assume $k=1$. Then [1, p. 280] there exist $a \geqq 1$ and $b > 0$ such that

$$|x(t; t_0, x_0)| \leqq a|x_0| \exp[-b(t - t_0)]$$

for all $t \geqq t_0 \geqq 0$. Thus we may choose $\beta_E(\delta) = a\delta$, $\alpha_E(\delta) = \frac{1}{2}\delta$, and $\tau(\delta) \equiv \tau = b^{-1} \log 2a$. Let $\beta_P(\delta) = (a+1)\delta$ and $\alpha_P(\delta) = 3\delta/4$. Then the right-hand side of (4.2) is a linear function of δ. Thus if $g(t, x) = o(|x|)$, (4.2) will be satisfied for sufficiently small δ.

Now let $k > 1$. Then [1, p. 279–80] there exist $a > 0$ and $b > 0$ such that

$$|x(t; t_0, x_0)| \leqq (a|x_0|^{1-k} + b(t - t_0))^{1/(1-k)}$$

for $t \geqq t_0 \geqq 0$ and there exist $c > 0$ and $T > 0$ such that

$$|x(t; t_0, x_0)| \leqq (|x_0|^{1-k} + c(t - t_0))^{1/(1-k)}$$

for $t_0 \geqq 0$ and $t \geqq t_0 + T$. Thus we may choose $\beta_E(\delta) = a_1\delta$, $\alpha_E(\delta) = \delta(1 - \delta^{k-1})^{1/(k-1)}$, and

$$\tau(\delta) \equiv \tau = 2^{k-1}(c(2^{k-1} - 1))^{-1} + T,$$

where $a_1 = a^{1/(1-k)}$. Then if $|x_0| \leqq \delta \leqq \frac{1}{2}$, $\tau \geqq (c(1 - \delta^{k-1}))^{-1}$; hence

$$|x(t_0 + \tau; t_0, x_0)| \leqq \alpha_E(\delta).$$

Let $\beta_P(\delta) = (a_1 + 1)\delta$ and $\alpha_P(\delta) = \frac{1}{2}(\delta + \alpha_E(\delta))$. Then the right-hand side of (4.2) becomes $q\delta\,[1 - (1 - \delta^{k-1})^{1/(k-1)}]$ for some constant $q > 0$. If this expression is divided by $[\beta_P(\delta)]^k$, its limit as $\delta \to 0$ is, using L'Hospital's rule, a positive constant. Thus (4.2) will be satisfied for sufficiently small δ provided that $\gamma(\beta_P(\delta))/[\beta_P(\delta)]^k \to 0$ as $\delta \to 0$, i.e., provided that $g(t, x) = o(|x|^k)$. This completes the proof.

References

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