# $D$-SEMIGROUPS

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If $G$ is a topological group, $G_0$ will denote the identity component.

DEFINITION 1. Let $\mathcal{C}$ denote the full subcategory of the category of locally compact abelian groups whose objects $G$ have the property that $G/G_0$ is a union of compact groups.

In [3] K. H. Hofmann described those locally compact semigroups which contain a proper dense maximal subgroup whose complement is compact and thus a group. Here we describe those locally compact semigroups which contain a dense subgroup $G \in \mathcal{C}$ whose complement is a group.

If $S$ is a topological semigroup, $E(S)$ will denote the set of idempotents of $S$ and $A^*$ will denote the closure of $A$ in $S$ where $A \subseteq S$. We use $R^n$ to denote the real $n$-dimensional vector group.

DEFINITION 2. A topological semigroup will be called a $D$-semigroup if $S$ satisfies the following hypotheses:

(i) $S$ is a locally compact Hausdorff semigroup.

(ii) $E(S)$ contains at least two elements 1 and $e$.

(iii) $H(1)^* = S$.

(iv) $eH(1)$ is a topological group.

DEFINITION 3. A Hausdorff semigroup $S$ will be said to be $H$-closed if $S$ contained in a Hausdorff semigroup $T$ as a subsemigroup implies $S$ is a closed subspace of $T$.

The following theorem is a result of K. H. Hofmann [3] along with the observation that a $D$-semigroup with $H(e)$ compact is $H$-closed [7].

THEOREM 1. *Let $S$ be a $D$-semigroup with $H(e)$ compact and $H(e) \cup H(1)$ a $D$-semigroup.*

*Then*

(i) $S = H(e) \cup H(1)$.

(ii) *Structure of $H(1)$. There is a maximal normal compact subgroup $C$ and a subgroup $M$, which is either a one-parameter group isomorphic to $R$ or an infinite cyclic group, and $H(1) = MC$, $M \cap C = \{1\}$.*

(iii) *The closure $M^*$ of $M$ is a disjoint union of $M$ and an abelian compact subgroup $A$ of $H(e)$ in which $Me$ is dense.*

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For $G\in\mathcal{C}$, $\hat{G}$ will denote the group of characters of $G$. For a morphism $i:G\to H$ in $\mathcal{C}$, $\hat{\imath}:\hat{H}\to\hat{G}$ will be the morphism defined by $\hat{\imath}(\alpha)=\alpha i$.

LEMMA 1. *For locally compact abelian groups the following statements are equivalent*:

(i) $G\in\mathcal{C}$.

(ii) $G\cong R^n\times K$ *where* $K$ *is a locally compact abelian group which is a union of compact subgroups.*

(iii) $(\hat{G})_0\cong R^n$.

PROOF. One has the general structure theorem [1, p. 389].

(I) A locally compact abelian group $G$ is topologically isomorphic with $R^n\times K$, where $K$ is a locally compact abelian group containing a compact open subgroup. If $G$ is topologically isomorphic with $R^m\times K_1$ and $K_1$ contains a compact open subgroup, then $m=n$.

(i)$\Leftrightarrow$(ii) If $G\cong R^n\times K$ where $K$ is a locally compact abelian group containing a compact open subgroup, then $G/G_0\cong K/K_0$ and $K_0$ is compact by (I). Thus $G/G_0$ is a union of compact groups if and only if $K/K_0$ is a union of compact groups if and only if $K$ is a union of compact groups.

(ii)$\Leftrightarrow$(iii) For an abelian locally compact group $K$, $K$ is a union of compact groups if and only if $\hat{K}$ is a totally disconnected [1, 383]. Thus $(\hat{G})_0\cong R^n$ if and only if $G\cong R^n\times K$ where $K$ is a union of compact subgroups.

LEMMA 2. *In the category of locally compact abelian groups* $\mathcal{C}$ *is closed under epics.*

PROOF. Let $i:H\to G$ be an epic in the category of locally compact abelian groups with $H\in\mathcal{C}$. Then $\hat{\imath}:\hat{G}\to\hat{H}$ is a monic, thus $\hat{\imath}$ is injective. Also, $\hat{\imath}[(\hat{G})_0]\subset(\hat{H})_0\cong R^n$. Therefore $(\hat{G})_0\cong R^m$ and $G\in\mathcal{C}$ by Lemma 1.

LEMMA 3. *Let* $i:G\to R^n$ *be an epic in* $\mathcal{C}$. *Then* $i$ *has a right inverse.*

PROOF. The morphism $\hat{\imath}:\hat{R}^n\to\hat{G}$ is monic, thus injective. Also, $\hat{R}^n\cong R^n$ and $i(\hat{R}^n)\subseteq(\hat{G})_0\cong R^m$. Thus $i(\hat{R}^n)$ splits in $(\hat{G})_0$, and $(\hat{G})_0$ splits in $\hat{G}$ by (I). Thus $\hat{\imath}$ has a left inverse, and $i$ has a right inverse.

LEMMA 4. *If* $i:G\to H$ *is an epic in* $\mathcal{C}$ *and* $G/G_0$ *is compact, then* $H/H_0$ *is compact.*

PROOF. By (I) $G\cong R^n\times K$ with a $K$ a compact group. Hence, $\hat{G}\cong R^n\times\hat{K}$ with $\hat{K}$ discrete [1, 362]. Since $\hat{\imath}$ is monic, $\hat{\imath}$ is injective. By (I) and since $\hat{\imath}^{-1}[(\hat{G})_0]$ is a closed subgroup of $\hat{H}$, $\hat{\imath}^{-1}[(\hat{G})_0]\cong R^p$

$\times K_1$ with $K_1$ totally disconnected. Since $\hat{\imath}$ is injective, $K_1$ is discrete. Since $(\hat{G})_0$ is open in $\hat{G}$ it follows that $(\hat{H})_0$ is open in $\hat{H}$. Thus $H \cong R^p \times C$ with $C$ a compact group.

If $S$ is a $D$-semigroup, $\phi$ will denote the map from $S$ to $eS$ defined by $\phi(s)=es$, $\Psi$ will denote the map from $H(1)$ to $H(e)$ defined by $\Psi(g)=eg$.

THEOREM 2. *Let $S$ be a $D$-semigroup with $H(1)\in\mathfrak{C}$. Then*

(i) *$H(e)\in\mathfrak{C}$.*

(ii) *$H(1)$ contains a locally compact subgroup $N$ and a vector subgroup $V$ such that the closure $(eN)^*_{H(e)}$ of $eN$ in $H(e)$ is a union of compact groups, $eV\cong V$, $V\cap N=\{1\}$, $H(1)=VN$, $eV\cap(eN)^*_{H(e)}=\{e\}$, $H(e)=V(eN)^*_{H(e)}$, and $N^*\cap H(e)=(eN)^*_{H(e)}$.*

(iii) *$H(1)$ contains a one-parameter group $P$ which is topologically isomorphic to $R$ and $(eP)^*$ is compact.*

PROOF. (i) Since $eH(1)\subseteq H(e)\subseteq(eH(1))$,* $H(e)$ is a topological group [2]; thus $H(e)$ is locally compact [8]. Since $H(1)^*=S$, the morphism $\Psi: H(1)\to H(e)$ is an epic; thus $H(e)\in\mathfrak{C}$ by Lemma 2.

(ii) By (I) and (i) there is a splitting surjection $\pi: H(e)\to W$ onto a vector group such that ker $\pi$ is a union of compact subgroups. The morphism $\pi\Psi: H(1)\to W$ is epic in $\mathfrak{C}$ and thus has a right inverse $i: W\to H(1)$ by Lemma 3. Let $V=i(W)$ and $N=\Psi^{-1}(\ker\pi)$. Let $r_1: H(1)\to V$ be the corestriction of $i\pi\Psi$ to its image; thus $r_1|V=1_V$. Since $s=r_1(s)[r_1(s)^{-1}s]$, $H(1)=VN$. Also, it follows that $eV\cong V$, $V\cap N=\{1\}$, $(eV)\cap(eN)^*_{H(e)}=\{e\}$, $H(e)=(eV)(eN)^*_{H(e)}$, and $(eN)^*\cap H(e)=\ker\pi$.

(iii) By (I) ker $\pi$ contains a compact open (in ker $\pi$) subgroup $C$. Then $\phi^{-1}(C)$ is a locally compact abelian semigroup which contains a compact kernel $C$. Thus there is an open subsemigroup $T$ of $\phi^{-1}(C)$ such that $C\subseteq T$ and $1\notin T^*$ [5, 115]. Thus if $g\in T\cap H(1)$, then $1\notin K(g)=\bigcap_{n=1}^{\infty}\{g^i \mid i\geq n\}^*$. If $\Psi^{-1}(C)$ is a union of compact groups, then for all $g\in\Psi^{-1}(C)$ $1\in K(g)$. Thus either $T\cap H(1)=\varnothing$ or $\Psi^{-1}(C)$ contains a one-parameter group isomorphic to $R$. Since $e\in(\Psi^{-1}(C))^*$, $T\cap H(1)\neq\varnothing$; thus $\Psi^{-1}(C)$ contains a one-parameter group $P$ isomorphic to $R$. By Weil's lemma [6, 102], $(eP)^*$ is compact.

COROLLARY. *Let $S$ be a locally compact Hausdorff semigroup which contains a dense group $H(1)\in\mathfrak{C}$. If $H(1)$ is a union of compact subgroups, then either $E(S)=\{1\}$ or for all $e\in E(S)\setminus\{1\}$, $H(e)$ is not a topological group.*

THEOREM 3. *Let $S$ be a $D$-semigroup with $H(1)\in\mathfrak{C}$ and $S\setminus H(1)=H(e)$. Then there is a vector subgroup $V\subseteq H(1)$ and a subsemigroup $T$ of $S$*

*such that*:

(i) *The function* $m: V\times T\to S$ *defined by* $m(v, t) = vt$ *is an isomorphism of topological semigroups.*

(ii) $T$ *is a D-semigroup with* $H_T(e) = T\backslash H_T(1)$.

(iii) $H_T(1) = H_S(1)\cap T \cong R\times K$ *where* $K$ *is a union of compact open subgroups of* $K$, *and* $H_T(e) = H_S(e)\cap T$ *is a union of compact open subgroups of* $H_T(e)$.

*If in addition,* $H_S(1)/H_S(1)_0$ *is compact, then* $T$ *is one of the semigroups described by Hofmann in* [**3**].

PROOF. Let $T = \phi^{-1}(\ker \pi)$. Then $T\cap H(e) = H_T(e) = \ker \pi$ is a union of compact open subgroups of $H_T(e)$. Let $r$ be the corestriction of $i\pi\phi$ to its image. Then $r| V = 1_V$. The morphism $s\to(r(s), r(s)^{-1}s)$: $S\to V\times T$ is the inverse of $m$ in (i). Part (ii) follows from Theorem 2. Finally, let $C$ be a compact open subgroup of $T\cap H(e)$. Then $\phi^{-1}(C)$ is open in $T$ and is a $D$-semigroup, but in addition $\phi^{-1}(C)\cap H(e)$ is compact. Thus by Hofmann's results [**3**], $\phi^{-1}(C)\cong R\times K$ with $K$ a compact group. Since $\phi^{-1}(C)\cap H(1)$ is open in $T\cap H(1)$, from (I) it follows that $T\cap H(1)\cong R\times K_1$ where $K_1$ is a union of compact subgroups. Since $H_T(e) = \ker \pi$, we have (iii).

Now assume $H(1)/H(1)_0$ is compact. Then $H_T(1)/H_T(1)_0$ is compact, hence, by Lemma 4 and (iii), $H(e)\cap T = H_T(e)$ is compact.

THEOREM 4. *Let* $S$ *be a D-semigroup with property that* $H(1)$ *is a topological group which contains a compact normal subgroup* $C$ *and* $H(1)/C\in\mathcal{C}$. *Then there is a D-semigroup* $S/C$ *and an open homomorphism* $h$ *from* $S$ *onto* $S/C$ *such that*:

(i) $h(H(1)) = H(h(1))\cong H(1)/C$.

(ii) $S/C = [H(h(1))]^*$.

(iii) $h[H(e)] = H(h(e))$, $h^{-1}[h(H(e))] = H(e)$, and $H(h(e))\cong H(e)/eC$.

The proof of this theorem presents no difficulties and is left to the reader.

COROLLARY 1. *Let* $S$ *be a locally compact Hausdorff semigroup satisfying the following hypotheses:*

(a) $E(S)$ *contains at least two elements* 1 *and* $e$.

(b) $H(1)$ *is a topological group such that* $H(1)^* = S$.

(c) $H(1)$ *contains a compact normal subgroup* $C$ *such that* $H(1)/C\cong R$.

*Then the boundary of* $H(1)$ *is a compact group if and only if* $eH(1)$ *is a topological group.*

PROOF. If the boundary of $H(1)$ is compact, then by the results of Hofmann [**3**] $S\backslash H(1)$ is a compact group.

If $eH(1)$ is a topological group, then $H(e)$ is a topological group.

By Theorem 4 $S/C$ is a $D$-semigroup with $h(H(1))\cong R$. By Theorem 2 $H(h(e))$ is compact; thus by a result of J. G. Horne [6], $S/C=H(h(1))\cup H(h(e))$. Thus $S=H(1)\cup H(e)$ by Theorem 3. Since $H(e)/eC$ is compact and $C$ is compact, $H(e)$ is compact.

COROLLARY 2. *Let $S$ be as in Theorem 4 with the additional property that $H(h(1))/H(h(1))$ is compact. Then $H(e)\cup H(1)$ is a $D$-semigroup if and only if $H(h(e))\cong R^{n-1}\times C$ where $C$ is compact and $H(h(1))\cong R^n \times K$ with $K$ compact.*

PROOF. Let $S/C$ be the semigroup described in Theorem 3. Then $H(e)\cup H(1)$ is a $D$-semigroup if and only if $H(h(e))\cup H(h(1))$ is a $D$-semigroup. If $H(h(e))\cup H(h(1))$ is a $D$-semigroup, then $h(H(e)\cup H(1))$ satisfies the conditions of Theorem 2. The only if part now follows from Theorem 2.

Assume $H(e)\cong R^{n-1}\times C$. By (I) there is a splitting surjection $\pi$ from $H(h(e))$ onto a vector group such that ker $\pi$ is compact. Then $\phi^{-1}(\ker \pi)$ is a $D$-semigroup with a compact ideal ker $\pi$ and $\phi^{-1}(\ker \pi)\cap H(h(1))\cong R\times K$ where $K$ is compact. Thus by Corollary 1 $\phi^{-1}(\ker \pi)=\ker \pi\cup(\phi^{-1}(\ker \pi)\cap H(h(1))$, and $\phi^{-1}(\ker \pi)$ is a closed subspace of $S/C$. By Lemma 3 the morphism $\pi\Psi\colon H(h(1))\to W$ has a right inverse $i\colon W\to H(h(1))$. Let $V=i(W)$ and $r\colon H(e)\cup H(1)\to V$ be the corestriction of $i\pi\phi$ to its image. Let $m\colon V\times\phi^{-1}(\ker \pi) \to h(H(1)\cup H(e))$ be the map defined by $m(v, t)=vt$. Then

$$s \to (r(s), r(s)s^{-1})\colon h(H(1) \cup H(e)) \to V \times \phi^{-1}(\ker \pi)$$

is the inverse of $m$. Thus $h(H(1)\cup H(e))$ is a $D$-semigroup.

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