A NONOSCILLATION THEOREM FOR A NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION

J. W. HEIDEL

In this paper we consider the real-valued solutions of the equation

$$y'' + q(t)y^{\gamma} = 0 \tag{1}$$

where $q(t) \geqq 0$ and continuous on some half line $[a, \infty)$ and γ satisfies $0<\gamma=p/q<1$ where p, q are odd integers. Our purpose is to give conditions under which all solutions of (1) are nonoscillatory. The result we give is similar to that given by Atkinson [**1**] for the case $\gamma>1$ but the proof is different.

The restriction to $\gamma=p/q$ where p and q are odd is significant. For example, if q is even and p odd, then oscillatory solutions are not real-valued. If p is even and q odd, then all nonzero solutions are trivially nonoscillatory. Similar problems arise if γ is irrational.

We begin with some definitions and basic facts. A solution of (1) is said to be extendable (continuable) if it exists on some half line $[b, \infty)$. Since $0<\gamma<1$, all solutions of (1) are extendable. This follows from a theorem of Wintner [**3**, p. 29]. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, a solution is called nonoscillatory, i.e., if it is of one sign for large t. Since γ is restricted to be odd, solutions with real initial conditions are real-valued and the negative of a solution is again a solution.

For the sake of completeness we state some related results. Ličko and Švec [**5**] have shown that all solutions of (1) are oscillatory if and only if $\int^{\infty} s^{\gamma} q(s)ds=\infty$. Belohorec [**2**] has shown the following. If there exists a number β, $0<\beta<(1-\gamma)/2$, such that $f(t)t^{(3+\gamma)/(2+\beta)} \uparrow K_1<\infty$, then all nontrivial solutions of (1) are nonoscillatory. If $f(t)t^{(3+\gamma)/2} \downarrow K_2>0$, then (1) has both (nontrivial) oscillatory and nonoscillatory solutions. For similar results pertaining to the case $1<\gamma$, see [**1**] and [**4**].

We can now state our major result. Its proof will be preceded by three lemmas.

THEOREM. *If $q(t) \in C'[a, \infty)$, $q(t)>0$ and $q'(t) \leqq 0$ for $t \geqq a$ and if $\int^{\infty} sq(s)ds<\infty$, then* (1) *has no oscillatory solutions.*

LEMMA 1. *Suppose that $\int^{\infty} sq(s)ds<\infty$ and let $K>0$ be given. Then*

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there is a $t_0 \geqq a$ *and a solution* $y(t)$ *of* (1) *defined on* $[t_0, \infty)$ *such that* $K/2 \leqq y(t) \leqq K$ *for* $t \geqq t_0$ *and* $\lim_{t\to\infty} y(t) = K$.

PROOF. Our proof is a modification of a proof given in [1]. Consider the integral equation

$$\psi(t) = K - \int_t^\infty (s-t)q(s)(\psi(s))^\gamma ds. \tag{2}$$

Let t_0 be such that

$$\int_{t_0}^\infty (s-t_0)q(s)ds < \min\{(K^{1-\gamma})/2,\ [\gamma(2/K)^{1-\gamma}]^{-1}\}.$$

To prove the lemma it suffices to show that (2) has a solution $\psi(t)$ such that $K/2 \leqq \psi(t) \leqq K$.

Let $\psi_0(t) \equiv K$, $t \geqq t_0$, and

$$\psi_{n+1}(t) = K - \int_t^\infty (s-t)q(s)(\psi_n(s))^\gamma ds, \qquad t \geqq t_0.$$

Then $K/2 \leqq \psi_n(t) \leqq K$ for $t \geqq t_0$. Note that $F(\psi) = \psi^\gamma$ satisfies a Lipschitz condition for $K/2 \leqq \psi \leqq K$ with Lipschitz constant $\gamma(2/K)^{1-\gamma}$. Therefore

$$|(\psi_{n+1}(t))^\gamma - (\psi_n(t))^\gamma| \leqq \gamma(2/K)^{1-\gamma}|\psi_{n+1}(t) - \psi_n(t)|$$

for $t \geqq t_0$ and

$$|\psi_{n+1}(t) - \psi_n(t)| \leqq \gamma(2/K)^{1-\gamma} \max_{t \geqq t_0} |\psi_n(t) - \psi_{n-1}(t)| \int_t^\infty (s-t)q(s)ds$$

also for $t \geqq t_0$. This shows that $\psi_n(t) \to \psi(t)$ uniformly on $[t_0, \infty)$ and hence $\psi(t)$ is a solution of (2) satisfying $K/2 \leqq \psi(t) \leqq K$ for $t \geqq t_0$.

LEMMA 2. *Suppose that* $q(t) \in C'[a, \infty)$, $q(t) > 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Let* $y(t)$ *be a nontrivial oscillatory solution of* (1). *Let* $\{t_n\}$ *be a sequence of consecutive relative maxima of* $|y(t)|$ *such that* $n > m \Rightarrow t_n > t_m$. *Then* $|y(t_n)|$ *is nondecreasing as* n *increases and* $\lim_{n\to\infty} t_n = \infty$.

PROOF. Multiply (1) by $y'(t)/q(t)$ and integrate from t_n to t_{n+1} to obtain

$$\int_{t_n}^{t_{n+1}} \frac{(y'(s))^2}{2} \cdot \frac{q'(s)}{(q(s))^2} ds + \frac{(y(t_{n+1}))^{\gamma+1}}{\gamma+1} - \frac{(y(t_n))^{\gamma+1}}{\gamma+1} = 0.$$

Since $q'(t) \leqq 0$, we get $|y(t_{n+1})| \geqq |y(t_n)|$.

Note that $\lim_{n\to\infty} t_n = \infty$ is not immediate because global uniqueness for initial value problems does not hold in the case $\gamma<1$. Suppose that $\lim_{m\to\infty} t_n = t^* < \infty$. Since $|y(t)|$ is increasing at its relative maxima, we can apply the mean value theorem to get a sequence $\{s_n\}\to t^*$ such that $\lim_{n\to\infty} |y'(s_n)| = \infty$. But this contradicts the fact that $y(t)$ exists on $[a, \infty)$.

LEMMA 3. *Let $u(t)$, $v(t)$, $w(t)$ be solutions of* (1) *satisfying $0\leqq u(t) \leqq v(t) \leqq w(t)$ for $t' \leqq t \leqq t''$. Define $\phi(t)$ by*

$$\phi(t) = (w - v)(v' - u') - (v - u)(w' - v').$$

Then $\phi(t') \geqq \phi(t'')$.

PROOF. The statement and proof of this lemma are adapted from Lemma 1 of [6]. Note that in our case

$$(v^\gamma - u^\gamma)(w - u) \geqq (w^\gamma - u^\gamma)(v - u).$$

PROOF OF THEOREM. Suppose to the contrary that $y_1(t)$ is an oscillatory solution of (1). Let $\{t_n\}$ be the sequence of consecutive relative maxima of $|y_1(t)|$. Then $\lim_{n\to\infty} t_n = \infty$ and $0 < \lim_{n\to\infty} |y_1(t_n)| \equiv L \leqq \infty$ by Lemma 2.

Let $0<K<L$ and let $y_2(t)$ be a solution of (1) such that $y_2(t)\uparrow K$ as $t\to\infty$ (by Lemma 1). Then we can find two points t', t'' such that the following situation occurs: $0<y_1(t') = y_2(t')$, $0<y_1(t'')=y_2(t'')$, and $0<y_2(t)<y_1(t)$ for $t'<t<t''$. If we now set $u(t)\equiv 0$, $v(t) = y_2(t)$, and $w(t) = y_1(t)$, we see that $\phi(t')<\phi(t'')$ ($\phi(t)$ is defined in Lemma 3). But this contradicts Lemma 3. This proves the theorem.

REMARK. The question arises as to whether the conditions $q(t)>0$, $q'(t)\leqq 0$ are necessary in the theorem. We conjecture that the weaker condition $q(t)\geqq 0$ is not sufficient. However, this weaker condition is sufficient for the following corollary.

COROLLARY. *If $\int^\infty sq(s)ds < \infty$ and if $y(t)$ is an oscillatory solution of* (1), *then* $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$.

PROOF. The fact that $\lim_{t\to\infty} y(t) = 0$ follows from the proof of the theorem. Given $\epsilon>0$ pick t_0 so large that $\int_{t_0}^\infty q(s)ds<1$ and $|y(t)|^\gamma<\epsilon$ for $t\geqq t_0$. Since $y(t)$ is oscillatory we may suppose that $y'(t_0)=0$. Therefore, by integrating (1) from t_0 to t we get

$$|y'(t)| \leqq \epsilon \int_{t_0}^{t} q(s) < \epsilon, \qquad t \geqq t_0.$$

Since ϵ is arbitrary it follows that $\lim_{t\to\infty} y'(t) = 0$ (see also [2, Theorem 2]).

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University of Tennessee