AN IDENTITY FOR THE SCHUR COMPLEMENT OF A MATRIX

DOUGLAS E. CRABTREE AND EMILIE V. HAYNSWORTH[1]

1. **Introduction.** Let $A=(a_{ij})$ be an $n\times n$ complex matrix, and suppose that B is a nonsingular principal submatrix of A. We define the *Schur complement of B in A*, denoted by (A/B), as follows: Let $\hat{A}$ be the matrix obtained from A by a simultaneous permutation of rows and columns which puts B into the upper left corner of $\hat{A}$.

$$A=\begin{bmatrix} B & E \\ D & G \end{bmatrix}.$$

Then $(A/B)=G-DB^{-1}E$.

Since

$$\det A=\det \hat{A}=\det\begin{bmatrix} I & 0 \\ -DB^{-1} & I \end{bmatrix}\det \hat{A}=\det\begin{bmatrix} I & 0 \\ -DB^{-1} & I \end{bmatrix}\begin{bmatrix} B & E \\ D & G \end{bmatrix}$$

$$=\det\begin{bmatrix} B & E \\ 0 & G-DB^{-1}E \end{bmatrix}=\det B\,\det(G-DB^{-1}E),$$

we see that

$$\det A = \det B\,\det(A/B).$$

This result is known as *Schur's formula.*

In case A is Hermitian, Haynsworth [5] has shown that the inertia of A can be determined from the inertia of any nonsingular principal submatrix of A together with that of its Schur complement. Other applications and properties of the Schur complement will appear in a later paper.

In §2 of this note, we prove that the Schur complement can also be constructed using quotients of minors of A. Details on this method of construction and its relation to partitioned matrices and M-matrices can be found in [1], [2], [3].

In §3, this construction is used to prove a quotient identity for the Schur complement: $(A/B)=((A/C)/(B/C))$.

2. **Elements of the Schur complement.** The notation $A(i_1,\cdots,i_p;j_1,\cdots,j_p)$ denotes the submatrix of A formed using rows $i_1,\cdots,i_p$

Received by the editors October 2, 1968.

[1] The work of the second author was done under Contract DA-91-591-EUC-3686 of the U. S. Army with the Institute of Mathematics, University of Basel. The author wishes to thank Professor A. M. Ostrowski for very helpful discussions.

and columns $j_1, \cdots, j_p$. For principal submatrices we abbreviate this notation to $A(i_1, \cdots, i_p)$.

Lemma. *Let $C=A(1, \cdots, k)$ be a nonsingular leading principal submatrix of A. Let $F=(f_{ij})$ be the matrix with elements*

$$f_{ij} = \det A(1, \cdots, k, i; 1, \cdots, k, j)/\det C \qquad (i, j = k+1, \cdots, n).$$

Then $F=(A/C)$, the Schur complement of C in A.

Proof. Let b_{ij} denote the bordered minor

$$b_{ij}=\det A(1, \cdots, k, i; 1, \cdots, k, j) = f_{ij} \det C.$$

With A partioned in the form

$$A = \begin{bmatrix} C & E \\ D & G \end{bmatrix},$$

let $D^{(i)}$ denote the $(i-k)$th row of D, and let $E_{(j)}$ denote the $(j-k)$th column of E. Thus

$$D^{(i)} = [a_{i1}, \cdots, a_{ik}] \qquad (i = k+1, \cdots, n)$$

and

$$E_{(j)} = \begin{bmatrix} a_{1j} \\ \cdot \\ \cdot \\ \cdot \\ a_{kj} \end{bmatrix} \qquad (j = k+1, \cdots, n).$$

Then for $i, j=k+1, \cdots, n$,

$$b_{ij} = \det \begin{bmatrix} C & E_{(j)} \\ D^{(i)} & a_{ij} \end{bmatrix}.$$

By Schur's formula, $b_{ij}=(a_{ij}-D^{(i)}C^{-1}E_{(j)})(\det C)$. Thus

$$f_{ij} = (a_{ij} - D^{(i)}C^{-1}E_{(j)}).$$

But these are precisely the elements of the matrix $(A/C)=G-DC^{-1}E$, so the lemma is proved.

We remark that the lemma allows us to restate a result contained in [1, Lemma 1]: *The Schur complement of an M-matrix is an M-matrix.*

3. The quotient property of the Schur complement.

Theorem. *If B is a nonsingular principal submatrix of A, and C*

is a nonsingular principal submatrix of B, *then* (B/C) *is a nonsingular principal submatrix of* (A/C), *and* $(A/B)=((A/C)/(B/C))$.

PROOF. We assume without loss of generality that $C=A(1, \cdots, k)$ and $B=A(1, \cdots, p)$, with $k<p$. Let $V=(A/C)$, of order $n-k$, and let $W=(B/C)$, of order $p-k$. (We label the rows and columns of V from $k+1$ to n. Similarly, the indices for W are $i, j=k+1, \cdots, p$, while for (V/W) we use $i, j=p+1, \cdots, n$.) It follows from the lemma that W is a principal submatrix of V. Moreover, W is nonsingular, since by Schur's formula,

$$\det W = (\det B)/(\det C).$$

Now let $\overline{V}=(\det C)V$. For $i, j=p+1, \cdots, n$ we have

$$\begin{aligned}(V/W)_{i,j} &= \det V(k+1, \cdots, p, i; k+1, \cdots, p, j)/\det W \\ &= \det C \det V(k+1, \cdots, p, i; k+1, \cdots, p, j)/\det B \\ &= \det \overline{V}(k+1, \cdots, p, i; k+1, \cdots, p, j)/\det B(\det C)^{p-k}.\end{aligned}$$

Since the elements of the matrix $\overline{V}$ are bordered minors from A, Sylvester's determinant identity [4] enables us to express the determinant of any square submatrix of $\overline{V}$ in terms of the corresponding submatrix of A. In particular,

$$\begin{aligned}&\det \overline{V}(k+1, \cdots, p, i; k+1, \cdots, p, j) \\ &\qquad = (\det C)^{p-k} \det A(1, \cdots, p, i; 1, \cdots, p, j).\end{aligned}$$

Thus

$$(V/W)_{i,j} = \det A(1, \cdots, p, i; 1, \cdots, p, j)/\det B,$$

which, by the lemma, equals $(A/B)_{i,j}$.

REFERENCES

1. Douglas E. Crabtree, *Applications of M-matrices to nonnegative matrices*, Duke Math. J. **33** (1966), 197–208.

2. ———, *Characteristic roots of M-matrices*, Proc. Amer. Math. Soc. **17** (1966), 1435–1439.

3. ———, *A matrix identity*, Amer. Math. Monthly **75** (1968), 648–649.

4. F. R. Gantmacher, *The theory of matrices*, Vol. 1, Chelsea, New York, 1959, p. 33.

5. Emilie V. Haynsworth, *Determination of the inertia of a partitioned Hermitian matrix*, Linear Algebra and Appl. 1 (1968), 73–81.

AMHERST COLLEGE AND
AUBURN UNIVERSITY