SUMMABILITY OF A CAUCHY PRODUCT SERIES

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1. Let $f(z)=\sum_{j=0}^{\infty} d_j z^j$ be analytic in the disk $|z|<R$, $R>1$. Let $d_j \geqq 0$ for $j=0, 1, 2, \cdots$, and let $f(1)=1$. The Sonnenschein matrix $D=(d_{nj})$, associated with $f(z)$ is defined by

$$[f(z)]^n = \sum_{j=0}^{\infty} d_{n_j} z^j, \qquad n = 1, 2, \cdots$$

$$[f(z)]^0 = 1.$$

Conditions guaranteeing the regularity of D have been given by Clunie and Vermes [**1**]. Let D' denote the transpose of D.

K. Ishiguro [**3**] established theorems of Abel's, Mertens' and Cauchy's type for the summability method (t_{n_k}) defined by

$$\begin{aligned} t_{n_k} &= \binom{k}{n} r^n(1-r)^{k-n} \qquad & k \geqq n \\ &= 0 & k < n. \end{aligned}$$

The matrix (t_{n_k}) is the transpose of the Euler matrix, (e_{n_k}), defined by

$$\begin{aligned} e_{n_k} &= \binom{n}{k} r^k(1-r)^{n-k} \qquad & k \leqq n \\ &= 0 & k > n. \end{aligned}$$

The Euler matrix is generated by the function $g(z)=1-r+rz$, [**4**]. It is the purpose of this note to establish theorems of Abel's, Mertens' and Cauchy's type for the transpose of a regular Sonnenschein matrix generated by the function $f(z)$.

2. Let the series, $\sum_{n=0}^{\infty} a_n x^n$, have radius of convergence equal to 1. The following notation is adopted from [**3**]. Let $a_n^*=\sum_{j=0}^{\infty} d_{j_n} a_j$. If $\sum_{n=0}^{\infty} a_n^* = A$, then we write

$$\sum_{n=0}^{\infty} a_n = A(D').$$

If $\sum_{n=0}^{\infty} a_n^*$ converges absolutely to A, then we write

Received by the editors February 20, 1969.

$$\sum_{n=0}^{\infty} a_n = A(|D'|).$$

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are given series, then we write

$$c_p = \sum_{m+n=p} a_m b_n \qquad p = 0, 1, 2, \cdots.$$

If $\sum_{n=0}^{\infty} (1/(n+1)) \sum_{j=0}^{n} a_j^* = A$, then we write

$$\sum_{j=0}^{\infty} a_j = A(D'; C, 1).$$

THEOREM 1. *If* $\sum_{n=0}^{\infty} a_n = A(D')$, $\sum_{n=}^{\infty} b_n = B(D')$ *and* $\sum_{n=0}^{\infty} c_n = C(D')$, *then* $AB = C$.

PROOF. Let z_0 be real and $|z_0| < 1$. Since $f(z)$ is analytic, the image of an interval, (α, β), about z_0 contains an interval, (u, v), about $f(z_0)$. If $(\alpha, \beta) \subset (-1, 1)$ then $(u, v) \subset (-1, 1)$ [**1**].

Let $q(x) = \sum_{n=0}^{\infty} a_n x^n$ and let $x = f(z)$ for $x \in (u, v)$ and $z \in (\alpha, \beta)$. Then

$$q(x) = \sum_{n=0}^{\infty} a_n [f(z)]^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} d_{n_k} z^k$$
$$= \sum_{k=0}^{\infty} z^k \sum_{n=0}^{\infty} d_{n_k} a_n = \sum_{k=0}^{\infty} \overset{*}{a}_k z^k.$$

The interchange in the order of summation is permissible since $d_{n_k} \geqq 0$ for $n = 0, 1, 2, \cdots, k = 0, 1, 2, \cdots$, and $\sum_{k=0}^{\infty} d_{n_k} |z|^k < 1$ if $|z| < 1$. Similarly we have

$$h(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{k=0}^{\infty} z^k \overset{*}{b}_k,$$

$$s(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} z^k \overset{*}{c}_k,$$

and $q(x)h(x) = s(x)$ for x close to $f(z_0)$ and z close to z_0. Thus

$$\overset{*}{c}_p = \sum_{m+n=p} \overset{*}{a}_n \overset{*}{b}_m.$$

The result now follows from Theorem 162 of [**2**].

The proof of Theorem 1 and Theorems 160, 161 and 164 of [**2**] yield the following results.

THEOREM 2. *If* $\sum_{n=0}^{\infty} a_n = A(|D'|)$ *and* $\sum_{n=0}^{\infty} b_n = B(|D'|)$, *then* $\sum_{n=0}^{\infty} c_n = C(|D'|)$ *and* $C = AB$.

THEOREM 3. *If* $\sum_{n=0}^{\infty} a_n = A(|D'|)$ *and* $\sum_{n=0}^{\infty} b_n = B(D')$, *then* $\sum_{n=0}^{\infty} c_n = C(D')$ *and* $C = AB$.

THEOREM 4. *If* $\sum_{n=0}^{\infty} a_n = A(\mathrm{D}')$ *and* $\sum_{n=0}^{\infty} b_n = B(D')$, *then* $\sum_{n=0}^{\infty} C_n = AB(D', C, 1)$.

It should be noted that if $f(0)=0$, it is sufficient to assume that $\sum_{n=0}^{\infty} a_n \mathrm{x}^n$ has radius of convergence $\rho > 0$. In this case we can restrict z so that $|f(z)| < \rho' < \rho$. Then the interchange in the order of summation in the proof of Theorem 1 is permissible.

REFERENCES

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