SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

TRACE AND THE CONVEX HULL OF THE SPECTRUM IN A VON NEUMANN ALGEBRA OF FINITE CLASS

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B. Fuglede and R. V. Kadison have shown that in a finite factor the (normalized) trace of an operator lies in the convex hull of its spectrum; the result appears as a corollary of determinant theory in such factors, and implies at once that a generalized nilpotent operator in such an algebra has trace zero [**4**, Theorem 2], [**3**, p. 108]. The aim of the present note is to extend this result to arbitrary von Neumann algebras of finite class, via an elementary proof that avoids determinant theory.

Let A be a C^*-algebra with unity, Σ the set of all normalized states of A. We write $\overline{W}(a)$ for the "closed numerical range" of $a \in A$; thus $\overline{W}(a) = \Sigma(a) = \{f(a) : f \in \Sigma\}$, and $\overline{W}(a)$ is the closure of the numerical range of a in any spatial representation of A [**1**, Theorems 2 and 3]. The following lemma is a theorem of S. Hildebrandt [**6**, Satz 4] in nonspatial form:

LEMMA. *If A is any C^*-algebra with unity, then*

$$\operatorname{conv} \sigma(a) = \bigcap \overline{W}(bab^{-1}) \tag{1}$$

for every $a \in A$, where $\sigma(a)$ is the spectrum of a, conv *denotes convex hull, and b runs over all invertible elements of A.*

PROOF. It suffices to observe that the constructions in Hildebrandt's Lemma 1—the uniformly convergent summation, inversion, and taking of positive square root—can be performed within A.

THEOREM. *If A is a von Neumann algebra of finite class, then for every $a \in A$ one has*

$$\overline{W}(a^\natural) \subset \operatorname{conv} \sigma(a), \tag{2}$$

that is,

$$\operatorname{conv} \sigma(a^\natural) \subset \operatorname{conv} \sigma(a), \tag{2'}$$

where $a^\natural$ denotes the normalized trace of a.

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PROOF. Fix $a \in A$ and let K_a be the norm-closed convex hull of the set $\{uau^*: u \in A \text{ unitary}\}$. If Z is the center of A, $K_a \cap Z = \{a^\natural\}$ by Dixmier's original construction of trace [2, Théorème 12], [3, p. 272, Corollaire]. Let $f \in \Sigma$. For $u \in A$ unitary, write $f_u(x) = f(uxu^*)$, $x \in A$; thus $f_u \in \Sigma$. If $x = \Sigma_1^n \lambda_i u_i a u_i^*$ is a convex combination of unitary transforms of a, one has $f(x) = (\Sigma_1^n \lambda_i f_{u_i})(a)$, where $\Sigma_1^n \lambda_i f_{u_i} \in \Sigma$, thus $f(x) \in \Sigma(a)$. Since f is continuous and $\Sigma(a)$ is closed, it follows that $f(K_a) \subset \Sigma(a) = \overline{W}(a)$. In particular, $f(a^\natural) \in \overline{W}(a)$; since $f \in \Sigma$ is arbitrary, we conclude that

$$(*) \qquad \overline{W}(a^\natural) \subset \overline{W}(a).$$

For any invertible $b \in A$ one has $(bab^{-1})^\natural = a^\natural$; from $(*)$ it follows that $\overline{W}(a^\natural) \subset \overline{W}(bab^{-1})$ for all invertible $b \in A$, and an application of (1) yields (2). Finally, since $a^\natural$ is normal it is convexoid, i.e. $\operatorname{conv} \sigma(a^\natural) = \overline{W}(a^\natural)$ (see e.g. [1, p. 502]).

COROLLARY. *In a von Neumann algebra of finite class, the trace of a generalized nilpotent element is zero.*

PROOF. In the notation of the theorem, if $\sigma(a) = \{0\}$ then $\overline{W}(a^\natural) = \{0\}$ and therefore $a^\natural = 0$. Incidentally, it follows that a is in the closed linear span of the commutators in A [2, Théorème 13].

Finally, we remark that the proof of the theorem (and its corollary) is valid for A an AW*-algebra of finite class possessing a trace (cf. [2], [5]); the question of the existence of a trace for a general AW*-algebra of finite class remains open, but the results apply, in any case, to the AW*-algebras of finite class constructed by F. B. Wright [7].

REFERENCES

1. S. K. Berberian and G. H. Orland, *On the closure of the numerical range of an operator*, Proc. Amer. Math. Soc. **18** (1967), 499–503.

2. J. Dixmier, *Les anneaux d'opérateurs de classe finie*, Ann. Sci. École Norm. Sup. 66 (1949), 209–261.

3. ———, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.

4. B. Fuglede and R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math (2) **55** (1952), 520–530.

5. M. Goldman, *Structure of AW*-algebras*. I, Duke Math. J. **23** (1956), 23–34.

6. S. Hildebrandt, *Über den numerischen Wertebereich eines Operators*, Math. Ann. **163** (1966), 230–247.

7. F. B. Wright, *A reduction for algebras of finite type*, Ann. of Math. (2) **60** (1954), 560–570.

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