ON THE COVERING OF POLYHEDRA BY POLYHEDRA[1]

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1. **Introduction.** A polyhedron P is the intersection of a finite number of closed half-spaces of a finite-dimensional Euclidean space. We do not exclude the possibility that P is empty or that P is the whole space. Suppose a polyhedron P is covered by a finite number of closed convex sets $C_1, \cdots, C_k$. Then it is intuitively plausible that, if one or more of the C_j is "round" (i.e., not a polyhedron), then each C_j could be replaced by a subpolyhedron and P would still be covered. The purpose of this note is to verify this intuition, and to show its relevance to a class of sufficient conditions for a rectangular matrix to be of full rank.

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2. THEOREM. *Let $C_1, \cdots, C_k$ be closed convex sets, P a polyhedron, $P=C_1\cup \cdots \cup C_k$. Then there exist polyhedra $P_j\subset C_j$, $j=1, \cdots, k$ such that $P=P_1\cup \cdots \cup P_k$.*

PROOF. We prove the theorem by induction on dim P (i.e., the dimension of the lowest dimensional flat containing P). The theorem obviously holds if dim $P=0$ or 1, so assume dim $P=n$ and that the theorem has been proved for all P such that dim $P<n$.

In the argument that follows we shall make use of the following well-known facts:

(2.1) The convex hull of a finite number of polyhedra is a polyhedron;

(2.2) If Q and R are polyhedra, the closure of $Q-R$ is the union of a finite number of polyhedra.

Our first step is to partition the space R^n containing P into closed orthants $Q_1, \cdots, Q_{2^n}$. Clearly each $P\cap Q_j=(C_1\cap Q_j)\cup \cdots \cup(C_k\cap Q_j)$, $j=1, \cdots, 2^n$. If we prove the theorem for each $P\cap Q_j$, then the theorem for P follows from (2.1). Hence we may assume P contained in a closed orthant, which we may take without loss of generality to be the first orthant. Therefore

$$(2.3) \quad x = (x_1, \cdots, x_n) \in P \quad \text{implies} \quad x_j \geqq 0, \qquad j = 1, \cdots, n.$$

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By the definition of dim P, P has nonempty interior. We use the symbol $|S|$ to denote the number of elements in the set S. For any covering $\mathcal{C}=\{C_1, \cdots, C_k\}$ of P we define the complexity $\mathcal{K}(\mathcal{C}, P)$ to be the ordered pair (r, s), where

$$r = \left| \{j \mid \text{Int } P \cap C_j \neq \varnothing\} \right|,$$
$$s = \left| \{S \subset \{1, \cdots, k\} \mid \text{Int } P \cap \bigcap_{j \in S} C_j \neq \varnothing\} \right|.$$

Further, we establish a linear order among the complexities of all coverings $\mathcal{C}$ of all n-dimensional polyhedra P by defining $\mathcal{K}(\mathcal{C}', P'') = (r', s') \prec \mathcal{K}(\mathcal{C}'', P'') = (r'', s'')$ if

$$r' < r'', \quad \text{or} \quad r' = r'', \qquad s' < s''.$$

Observe that $r=1$ implies $s=1$. For $\mathcal{K}(\mathcal{C}, P)=(1, 1)$, the theorem is obvious, for if Int $P \subset C_j$ for some j, then $P=C_j$ because C_j is closed. We now proceed by induction on $\mathcal{K}(\mathcal{C}, P)$.

Assume that $\mathcal{K}(\mathcal{C}, P)=(r, s)$, and the theorem has been proved for all preceding complexities. Clearly $r \leqq s \leqq 2^r-1$, $r>1$.

Case 1. Assume $s<2^r-1$. Then there exists a nonempty subset $S \subset \{1, \cdots, k\}$ and an index m such that the convex sets $K=\text{Int } P \cap \bigcap_{j \in S} C_j \neq \varnothing$ and $L=\text{Int } P \cap C_m \neq \varnothing$ satisfy $K \cap L=\varnothing$. They can therefore be weakly separated; i.e., there exists a nonzero vector a and a constant b such that

$$H_K = \{x \mid (a, x) > b\} \quad \text{and} \quad H_L = \{x \mid (a, x) < b\} \tag{2.3}$$

satisfy $H_K \cap K=\varnothing$, $H_L \cap L=\varnothing$.

Let $P_K=P \cap$closure of H_K, $P_L=P\cap$closure of H_L, $\mathcal{C}_K=\{C_1 \cap P_K, \cdots, C_k \cap P_K\}$, $\mathcal{C}_L=\{C_1\cap P_L, \cdots, C_k \cap P_L\}$. Let $(r, s)=\mathcal{K}(\mathcal{C}, P)$, $(r^K, s^K)=\mathcal{K}(\mathcal{C}_K, P_K)$, $(r^L, s^L)=\mathcal{K}(\mathcal{C}_L, P_L)$. From (2.3), $r^K \leqq r$, $s^K<s$ (since Int $P \cap \bigcap_{j \in S} C_j \neq \varnothing$, but Int $P_K \cap \bigcap_{j \in S} (C_j \cap P_k)=\varnothing$)), and $r^L<r$ (since Int $P \cap C_m \neq \varnothing$, but Int $P_L \cap (C_m \cap P_L)=\varnothing$). It follows from the induction hypothesis on complexity that there exist polyhedra $P_{Kj} \subset C_j \cap P_K \subset C_j$, $j=1, \cdots, k$ and $P_{Lj} \subset C_j \cap P_L \subset C_j$, $j=1, \cdots, k$ such that $P_K=P_{K1} \cup \cdots \cup P_{Kk}$, $P_L=P_{L1} \cup \cdots \cup P_{Lk}$. The theorem now follows from (2.1).

Case 2. Assume $s=2^r-1$. Without loss of generality, we may assume $r=k$, and it follows that $D=\bigcap_{j=1}^k C_j \neq \varnothing$.

Case 2a. Assume D bounded. Let E be any bounded polyhedron (if P is bounded, take $P=E$) such that $D \subset E$, and let x be an arbitrary point in D. Then $P \cap E$ is a bounded polyhedron with faces $F_1, \cdots, F_f$. Clearly each $F_i=\bigcup_{j=1}^k F_i \cap C_j$. Since dim $F_i<n$, our induction hypothesis on the dimension shows that for each i there

exist polyhedra P_{ij} such that $P_{ij} \subset F_i \cap C_j \subset C_j$, $j=1, \cdots, k$, with $F_i = \bigcup_{j=1}^{k} P_{ij}$. Because $P \cap E$ is bounded, each point of $P \cap E$ is on a line segment joining x to a face of $P \cap E$. Let P_j be the convex hull of x and all P_{ij}, $i=1, \cdots, f$. Then $P \cap E = \bigcup_{j=1}^{k} P_j$ and $P_j \subset C_j$, $j=1, \cdots, n$.

Now the closure of $P-(P \cap E)$ is by (2.2) the union of polyhedra, say $Q_1, \cdots, Q_q$. Each Q_i is covered by $\{Q_i \cap C_1, \cdots, Q_i \cap C_k\}$, and the complexity of each such covering of each Q_i precedes $\mathcal{K}(\mathcal{C}, P)$, since $D \cap \text{Int } Q_i = \varnothing$ for each i. Thus the induction hypothesis on complexity proves the theorem for each Q_i and we now apply (2.1).

Case 2b. Assume D unbounded. Then D contains a ray i.e., there is a point x and a vector t such that $x+\lambda t \in D$ for all $\lambda \geqq 0$. Call this ray R and note that R is a polyhedron contained in each C_j. Since $D \subset P$ and P is in the first orthant, it follows that

(2.4) $\quad x_i \geqq 0 \qquad i = 1, \cdots, n.$

(2.5) $\quad t_i \geqq 0 \qquad i = 1, \cdots, n, \qquad$ at least one $t_i > 0$.

(2.6) $\quad$ bdry P is nonempty.

By the induction hypothesis on dimension, if $F_1, \cdots, F_f$ are the faces of P, there exist polyhedra P_{ij}, $i=1, \cdots, f; j=1, \cdots, k$, such that $P_{ij} \subset C_j$, $j=1, \cdots, k$, $\bigcup_{j=1}^{k} P_{ij} = F_i$. Let P_j be the convex hull of R and all P_{ij}, $i=1, \cdots, n$. Clearly $P_j \subset C_j$, and all we need show is that every point of P is on a line segment joining a point of R with a point on bdry P.

Let $y=(y_1, \cdots, y_n) \in P$. From (2.5) there exists a point $r=(r_1, \cdots, r_n) \in R$ and an index i_0 such that $r_{i_0} > y_{i_0}$.

Consider the point $y(\alpha) = (1/(1-\alpha))(y_1-\alpha r_1, \cdots, y_n - \alpha r_n)$, which is a continuous function of α. When $\alpha=0$, $y(\alpha)=y$. When α is close to 1, since $r_{i_0} > y_{i_0}$, $y(\alpha)$ is outside the first orthant, hence outside P. Hence there is a number α_0, $0 \leqq \alpha_0 < 1$ such that $y(\alpha_0) \in$ bdry P. Thus

$$y = (1-\alpha_0) y(\alpha_0) + \alpha_0 r,$$

which is the result desired.

3. **An application.** Let $n > m$, and let $C_1, \cdots, C_n$ be closed convex pointed cones in R^n. For any $n \times m$ real matrix A, denote the rows of A by $A_1', \cdots, A_n'$. For any cone C and any vector x, the expression $x'C > 0$ means that x makes a positive inner product with every nonzero $y \in C$. In an investigation of systems of strong linear inequalities on the rows of an $n \times m$ matrix which imply that the matrix is of full rank, the equivalence of the following two statements was proved [1]:

$$(3.1) \qquad A_i' C_i > 0 \quad \text{for all } i \text{ implies rank } A = m$$

$$(3.2) \qquad \bigcup_{i=1}^{n} C_i \cup \bigcup_{i=1}^{n} - C_i = R^m.$$

Because of the potential usefulness of (3.1) in numerical work, it is of some interest to find families of cones $\{C_i\}$ satisfying (3.2) such that if cones $D_i \subset C_i$ also satisfy (3.2) then $D_i = C_i$. The reason is that "smaller" cones prove more matrices to be of full rank. That such minimal cones exist is an easy consequence of Zorn's lemma, as Alex Heller has kindly pointed out to us (partial order the families of cones satisfying (3.2) by respective inclusion, choose a maximal simply ordered subset and form intersections). We now show that such a minimal family of cones is polyhedral. Assume $\{C_i\}$ satisfying (3.2). Since $\bigcup C_i \cup \bigcup - C_i = R^m$, and R^m is a polyhedron, there exist polyhedra P_i, $i = \pm 1, \cdots, \pm n$, such that $P_i \subset C_i$, $i = 1, \cdots, n$, $P_{-i} \subset -C_i$, $i = -1, \cdots, -n$, and $\bigcup_{i=1}^{n} P_i \cup \bigcup_{i=-1}^{-n} P_i = R^m$. Let E_i be the polyhedral cone spanned by P_i and $-P_{-i}$, $i = 1, \cdots, n$. Then $E_i \subset C_i$, $i = 1, \cdots, n$ and $\bigcup E_i \cup \bigcup - E_i = R^m$, since $P_i \subset E_i$, $P_{-1} \subset -E_i$, $i = 1, \cdots, n$.

4. **Remark.** We conjecture (but have been unable to prove) the following: Let $C_1, \cdots, C_k$ be closed convex subsets of a bounded polyhedron P, and let $0 < t$ be an integer such that every t-flat which meets P meets $\bigcup_{i=1}^{k} C_i$; then there exist polyhedra $P_i \subset C_i$, $i = 1, \cdots, k$ such that every t-flat which meets P meets $\bigcup P_i$.

References

1. A. J. Hoffman, *On unions and intersections of cones*, Proceedings of the Third Waterloo Conference on Combinatorial Mathematics (to appear).

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