A GRONWALL INEQUALITY FOR LINEAR STIELTJES INTEGRALS

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This paper provides a Gronwall type inequality which includes the one found by Schmaedeke and Sell [**4**].

Suppose that S is an interval of real numbers containing zero and OB is the collection of functions from S to the real numbers each member of which is of bounded variation on each finite interval of S. The numeral 1 will also denote the constant function from S which has only the value 1; if x is in S, then 1_x denotes the function from S which has the value 1 at x and the value 0 elsewhere; and 0_x denotes the function $1-1_x$. Let J be a function from OB to the collection of functions from $S\times S$ to the real numbers having the following properties: if each of f and g is in OB and $\{x, y, z\}$ is in $S\times S\times S$ then

(1) $J[f](x, y)+J[g](x, y)=J[f+g](x, y)$,

(2) if r is a number then $J[r\cdot f](x, y)=r\cdot J[f](x, y)$,

(3) $J[f](x, y)+J[f](y, z)=J[f](x, z)$ provided that $x\leqq y\leqq z$ or $x\geqq y\geqq z$,

(4) $J[f](x, z)\geqq 0$ provided that $f(y)\geqq 0$ for $x\leqq y\leqq z$ or $x\geqq y\geqq z$, and

(5) if x is in S and $x\geqq 0$ then each of $J[0_x](x, x^+)$ and $J[1_x](x^-, x)$ is less than 1; whereas, if x is in S and $x\leqq 0$ then each of $J[1_x](x^+, x)$ and $J[0_x](x, x^-)$ is less than 1.

THEOREM. *If J satisfies properties* (1)–(5), *there is a function m from $S\times S$ to the real numbers having the following properties*:

(i) $m(x, y)\geqq 1$ *for each* $\{x, y\}$ *in* $S\times S$,

(ii) $m(x, y)\cdot m(y, z)=m(x, z)$ *provided that* $x\leqq y\leqq z$ *or* $x\geqq y\geqq z$,

(iii) $m(0, x)=1+J[m(0, \cdot)](0, x)$ *for each x in S, and*

(iv) *if f is in OB, P is a number, and* $f(x)\leqq P+J[f](0, x)$ *for each x in S, then* $f(x)\leqq P\, m(0, x)$ *for each x in S.*

REMARK. It is the purpose of this remark to show a connection between the above theorem and one of Schmaedeke and Sell. In [**4**], they investigate an inequality similar to that in part (iv) but use the mean Stieltjes integral and the Dushnik or interior integral (see also [**3**]). One-term approximating sums for these are indicated:

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$$(\mathrm{M}) \int_x^y f dg \sim \frac{f(x) + f(y)}{2} \cdot [g(y) - g(x)]$$

and

$$(\mathrm{I}) \int_x^y f dg \sim f(z) \cdot [g(y) - g(x)] \quad \text{where} \quad x < z < y \quad \text{or} \quad x > z > y.$$

If no member of S is negative, g is increasing and right continuous, and, for $x \leqq y$, $J[f](x, y)$ is defined to be (M) $\int_x^y f\, dg$, with $J[f](y, x) = J[f](x, y)$, then J satisfies properties (1)–(4) and, also, property (5) in case $g(z) - g(z^-) < 2$ for all z different from zero. If P is a number, f is in OB, and $f(x) \leqq P + (\mathrm{M}) \int_0^x f\, dg$ for each x in S, then $f(x) \leqq P + J[f](0, x)$ for each x in S, since no member of S is negative. This inequality includes the inequality of [**4**, p. 1219]. If, instead, J is defined in terms of the interior integral then properties (1)–(4) are, again, satisfied by J and property (5) makes no additional requirement due to the condition that g is right continuous. (See Remark 1 of [**4**].)

REMARK. With properties (1)–(3), a more familiar property which is equivalent to the conceptually simpler property (4) is

(4′) if f is in OB and $\{x, z\}$ is in $S \times S$ and m is a number such that $|f(y)| \leqq m$ for all y in S such that $x \leqq y \leqq z$ or $x \geqq y \geqq z$ then $|J[f](x, z)| \leqq mJ[1](x, z)$ (compare [**2**, Axiom II]).

To see that (4) implies (4′), notice that each of $m + f(y)$ and $m - f(y)$ is nonnegative for $x \leqq y \leqq z$ or $x \geqq y \geqq z$; to see that (4′) implies (4), notice that each of $J[1_x]$ and $J[0_x]$ has only nonnegative values and use the formulas in Theorem 1 and equation (24) of [**2**]. We shall use the fact that if f is in OB and $\{x, y\}$ is in $S \times S$ then $|J[f](x, y)| \leqq J[\,|f|\,](x, y)$ which follows from properties (1)–(4).

INDICATION OF PROOF OF THEOREM. The proof of parts (i), (ii), and (iii) of the theorem is only a slight modification of the ideas developed by MacNerney in [**2**, Theorems 1 and 2] and used by the author in [**1**, Theorem 1.1]. For part (iv), suppose that f is in OB, P is a number, and $f(x) \leqq P + J[f](0, x)$ for each x in S. Define a sequence h with values in OB as follows: $h_0 = f$ and, if n is a positive integer, then $h_n(x) = P + J[h_{n-1}](0, x)$ for each x in S. Let r be a function so that if x is in S then $r(x) = \int_0^x |d[h_2 - h_1]|$. Let L be a sequence so that if x is in S then $L_1(x) = r(x)$ and if n is a positive integer then $L_{n+1}(x) = J[L_n](0, x)$. If n is a positive integer and a is in S, then

$$0 \leqq \sum_{p=1}^{n} L_p(a) \leqq \sum_{p=1}^{n+1} L_p(a) \leqq r(a) \cdot m(0, a).$$

Moreover, if x is in S and between 0 and a and n is a positive integer then $L_n(x) \leqq L_n(a)$. Finally, if n is a positive integer and x is in S then $|h_{n+1}(x) - h_n(x)| \leqq L_n(x)$. Thus the sequence h converges absolutely and, if a is in S, uniformly on the set of all numbers in S between 0 and a. Moreover, if $\lim h = U$ and a is in S, then $U(a) = P + J[U](0, a)$; and $U(x) = Pm(0, x)$ for each x in S. (To see this latter, recall [2, Theorems 2 and E].) We have, inductively, that if p is a positive integer and x is in S then $f(x) \leqq h_p(x)$. Consequently, $f(x) \leqq Pm(0, x)$.

REMARK. Using [1, Lemma 1.1] and similar techniques to the ones indicated above, we may obtain a more general inequality for a function f which satisfies $f(x) \leqq P + J[f](0, x) + g(x)$ where g is in OB and $g(0) = 0$.

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