PSEUDO-UNIFORM CONVEXITY IN H^1

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1. **Introduction.** The following theorem is due to D. J. Newman [3]: if f_n, f are in H^1 of the unit circle, $\|f_n\|\to\|f\|_1$ and $f_n(z)\to f(z)$ uniformly on compact subsets of $|z|<1$, then $\int|f_n-f|\,d\theta\to 0$. Newman refers to this property of H^1 as pseudo-uniform convexity.

Let A be a Dirichlet algebra on the compact Hausdorff space X and let $H^1(dm)$ be the closure of A in $L^1(dm)$ where m is a nonnegative finite Borel measure on X such that $f\hat{\ }(0)=\int_X f\,dm$ defines a multiplicative linear functional on A. For the necessary results concerning Dirichlet algebras, see [2, p. 54]. In some work on invariant subspaces [4], the question arose as whether or not $H^1(dm)$ was pseudo-uniformly convex. It was shown there that if f_n is a sequence in $H^1(dm)$ such that $\|f_n\|_1\leqq 1$ and $\int_X f_n\,dm\to 1$, then $\|f_n-1\|_1\to 0$. See also Bull. Amer. Math. Soc. **71** (1965), p. 855.

The purpose of this paper is to present an argument for Newman's theorem which is independent of the interior of the unit circle and to give an example which is a result of our attempts to generalize the argument to the torus. The example shows that $H^1(dm)$ need not be pseudo-uniformly convex. We still do not know necessary and sufficient conditions in order that $H^1(dm)$ should be pseudo-uniformly convex.

2. **Newman's Theorem.**

THEOREM. *If f_n and f are in H^1 of the unit circle such that $\|f_n\|_1\to\|f\|_1$ and $f_n(z)\to f(z)$ uniformly on compact subsets of $|z|<1$, then $\|f_n-f\|_1\to 0$.*

PROOF. Without loss of generality, we may assume that $\|f_n\|_1=\|f\|_1=1$ and that $\int f\,d\theta\neq 0$. Each f_n has a factorization $f_n=g_nh_n$, where g_n and h_n are in H^2 and $|g_n|^2=|h_n|^2=|f_n|$ almost everywhere with respect to $d\theta$, [2, p. 71]. Since the unit ball of H^2 is weakly compact, there exist g and h in H^2 such that $\|g\|_2\leqq 1$, $\|h\|_2\leqq 1$ and g and h are weak limit points of $\{g_n\}$ and $\{h_n\}$ respectively. By passing to subsequences, we may assume that $g_n\to g$ and $h_n\to h$ weakly. For p a nonnegative integer, we have

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$$\hat{g_n}(p) \to \hat{g}(p) \quad \text{and} \quad \hat{h_n}(p) \to \hat{h}(p),$$

where $\phi\hat{}$, $(\phi \in L^1)$, denotes the Fourier transform of ϕ.

Since

$$(g_n h_n)\hat{}(p) = \sum_{i=0}^{p} \hat{g_n}(p-i)\hat{h_n}(i),$$

we have $\hat{f_n}(p) = (g_n h_n)\hat{}(p) \to (gh)\hat{}(p)$. On the other hand, the hypothesis that $f_n(z) \to f(z)$ uniformly on compact subsets of $|z| < 1$ is equivalent to $\hat{f_n}(p) \to f\hat{}(p)$ (p an integer), [1]. Thus $f = gh$ and from the inequalities

$$1 = \int |f|\, d\theta = \int |gh|\, d\theta \leqq \|g\|_2 \|h\|_2 \leqq 1,$$

we conclude that $\|g\|_2 = \|h\|_2 = 1$. The uniform convexity of H^2 implies that

$$\|g_n - g\|_2 \to 0 \quad \text{and} \quad \|h_n - h\|_2 \to 0.$$

It is now clear that $\|f_n - f\|_1 \to 0$ as required.

3. **An example.** Let T^2 denote the 2-dimensional torus and let Z^2 denote the set of lattice points in the plane. We shall regard Z^2 as the group of continuous characters on T^2. We define an order on Z^2 by

$$(r, s) < (p, q) \qquad \text{if } r < p \quad \text{or if} \quad r = p \quad \text{and} \quad s < q.$$

If P is the set of nonnegative lattice points (with respect to the above order), then P is a half-plane, see [2, p. 54]. The algebra of all continuous functions whose Fourier transforms vanish outside P is a Dirichlet algebra on T^2. Also, $H^p(dm)$, $1 \leqq p < \infty$, is the set of all functions in L^p of the torus whose Fourier transforms are zero in the complement of P.

In order to simplify the notation, let

$$\theta(x, y) = e^{ix} \quad \text{and} \quad \phi(x, y) = e^{iy},$$

for $(x, y) \in T^2$. If n is a positive integer, we define g_n by

$$g_n = \frac{1}{2} \sum_{k=0}^{\infty} 2^{-k/2}(\phi^{k+n} + \theta\phi^{-k-n}).$$

Each g_n is clearly continuous on T^2 and $\|g_n\|_2 = 1$. Also, if $\hat{g_n}(p, q) \neq 0$, then $p = 0$ and $q \geqq n$ or $p = 1$ and $q \leqq -n$ and thus $g_n \in H^2(dm)$. As before, $\hat{g_n}$ denotes the Fourier transform of g_n. We now let $f_n = g_n^2$ and note that $f_n \in H^1(dm)$ and that $\|f_n\|_1 = 1$. A straightforward computation shows that

$$f_n = \sum_{k=-\infty}^{\infty} 2^{-|k|/2}\theta\phi^k + \frac{1}{4}\sum_{k=0}^{\infty} 2^{-k/2}(k+1)(\phi^{2n+k} + \theta^2\phi^{-2n-k}).$$

Consider the function f defined by

$$f = \sum_{k=-\infty}^{\infty} 2^{-|k|/2}\theta\phi^k.$$

It is clear that $\hat{f_n} \to f\hat{}$ pointwise and that $f_n \to f$ weakly. Since

$$f = \frac{1}{2}\left(\sum_{k=0}^{\infty} 2^{-k/2}\phi^k\right)\left(\sum_{k=0}^{\infty} 2^{-k/2}\phi^{-k}\right)\theta$$

and $1 = |f\hat{}(1, 0)| \leqq \|f\|_1$, an application of the Schwarz inequality shows that $\|f\|_1 = 1$. However, f is not the norm limit of $\{f_n\}$ since

$$\|f - f_n\|_1 \geqq |\hat{f}(0, 2n) - \hat{f_n}(0, 2n)| = \tfrac{1}{4}.$$

The example is now complete.

References

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