UPPER BOUNDS ON THE DIMENSION OF EXTENDIBILITY OF SUBMANIFOLDS IN C^n

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1. **Introduction.** Suppose K is a subset of C^n. Then $\mathcal{H}(K)$ is the collection of all functions holomorphic in a neighborhood of K. We say that K is extendible to a connected set K' of C^n if $K \subsetneq K'$, and the natural restriction map from $\mathcal{H}(K')$ to $\mathcal{H}(K)$ is onto.

In the special case that K is a submanifold, M, of C^n it is interesting to ask for "geometric" conditions on M that insure results on extendibility. Such results can be proven for C-R submanifolds of C^n. These manifolds also have interesting applications to partial differential equations. (See [1], [2], for details of what follows.)

$T(C^n)\otimes C$ has a splitting into two equal-dimensional subbundles, $H(C^n)$ and $A(C^n)$, obtained from the complex structure of C^n. $H(C^n)_p$ is generated by $\partial/\partial z_j|_p$, $1\leqq j\leqq n$, and is called the holomorphic tangent bundle of C^n. $A(C^n)$, the antiholomorphic tangent bundle, is the conjugate of $H(C^n)$ in $T(C^n)\otimes C$.

If M is a differentiable submanifold of C^n, M is called a C-R submanifold of C^n if $H(M)=T(M)\otimes C\cap H(C^n)$ is a vector bundle. If M is a C-R manifold, then $A(M)=T(M)\otimes C\cap A(C^n)$ is also a vector bundle. $H(M)\cap A(M)=0$, and $H(M)$ (resp. $A(M)$) is involutive. The Levi algebra of M, $\mathcal{L}(M)$, is the Lie subalgebra of complex vector fields generated by sections of $A(M)$ and $H(M)$. We make the *assumption* that the dimension of $\mathcal{L}(M)$ is constant. Then $\mathcal{L}(M)$ is the algebra of sections of a vector bundle V, and $V\supset H(M)+A(M)$. Let $e=$ fiber $\dim_C V/(H(M)+A(M))$. e is called the excess dimension of $\mathcal{L}(M)$, ex dim $\mathcal{L}(M)$.

Now max $(\dim M-n, 0)\leqq$ fiber $\dim_C H(M)\leqq n$. If fiber $\dim_C H(M)$ $=$ max $(\dim M-n, 0)$, M is called generic. There are two results:

THEOREM (NIRENBERG AND WELLS [2]). *If M is a compact generic C-R submanifold of C^n, and* $\dim M\leqq n$, *then M is not extendible.*

THEOREM ([1]). *If M is a generic C-R submanifold of C^n, and* $\dim M>n$, *then M is locally extendible to a set containing a differentiable manifold N, with* $\dim N=\dim M+e$. *If $e=0$, then M is locally holomorphically convex.*

Received by the editors May 23, 1968.

(We say a set is locally extendible if each sufficiently small open subset of it is extendible. A set K is locally holomorphically convex if, for any $p \in K$, $K \cap B$ is not extendible, for B a sufficiently small open ball in C^n centered at p.)

In the following section we prove a stronger version of the second theorem above for real analytic C-R submanifolds of C^n, by removing the restriction "generic" and establishing an upper bound on the dimension of local extendibility. We comment on the possibility of proving the theorem for differentiable C-R submanifolds.

Conversations with Professor Hugo Rossi during the preparation of this paper proved invaluable.

2. **Real analytic *C-R* submanifolds.** If M is a C-R submanifold of C^n, then the C-R codimension of M, C-R codim M, is dim M—fiber dim $H(M)$. If M is generic and dim $M \geqq n$, then codim M in $C^n = C$-R codim M.

If $(M, H(M))$ and $(M', H(M'))$ are C-R manifolds, a C-R map $f: M \to M'$ is a differentiable map so that $df(H(M)) \subset H(M')$. If M is a C-R submanifold of C^n, the restriction of any element of $\mathcal{H}(M)$ to M is a C-R map from M to C.

THEOREM. *Let M be a nontrivial real analytic C-R submanifold of C^n (so $H(M) \neq 0$). If $e=0$, M is locally holomorphically convex. If $e>0$, M is locally extendible to a set containing a manifold N, with* dim $N=$dim $M+e$. *M is not locally extendible to a set of dimension greater than* dim $M+e$.

PROOF. Suppose M is a real analytic C-R submanifold of C^n, and dim $M=k$, and C-R codim $M=l$. Then, if we select $p \in M$, we can find $m(=\frac{1}{2}(k+l))$ linear combinations $S_1, \ldots, S_m$ of the coordinate functions $z_1, \ldots, z_n$ so that the map $S: C^n \to C^m$ given by $S=(S_1, \ldots, S_m)$ imbeds M near p as a generic C-R submanifold of C^m. We restrict our attention to that part of M which is imbedded generically. Then (using a complexification argument due to Tomassini [3]) any real analytic C-R map $f: M \to C$ is the restriction of a holomorphic function defined in a neighborhood of $S(M)$ in C^m.

If we suppose ex dim $\mathcal{L}(M)=e>0$, then ex dim $\mathcal{L}(S(M))=e$. We must show that M is extendible to a set L containing an $(e+k)$ dimensional manifold. If $f \in \mathcal{H}(M)$, then $f|_M: M \to C$ is a C-R map. So there is $f^* \in \mathcal{H}(S(M))$ with $f^*|_{S(M)} = f|_M \circ S^{-1}|_{S(M)}$. f^* extends to a set L^* (since $S(M)$ is generic) and L^* contains an $(e+k)$ dimensional manifold. Consider now the functions $z_1, \ldots, z_n \in \mathcal{H}(M)$. There are associated $z_1^*, \ldots, z_n^* \in \mathcal{H}(S(M))$ which extend to L^*. We define $L: P$

$=(p_1, \ldots, p_n) \in L$ when there is $q \in L^*$ with $p_j = z_j^*(q)$, $1 \leq j \leq n$. By the way we constructed the map S, we see that L must contain an $(e+k)$ dimensional manifold since L^* does. We define an extension of f to L by taking the extension of f^* and transporting back to L. By the way L was constructed this is insured to be an analytic function of $z_1, \ldots, z_n$.

If $e=0$, a similar argument will show that M is locally holomorphically convex. Or, there is also a simple complexification argument for this case.

To complete the remaining assertions of the theorem, we show that there is a real analytic C-R manifold, N, with dim $N=(k+e)$, and ex dim $\mathfrak{L}(N)=0$, so that

$$\begin{array}{ccc} M & \xrightarrow{\;\;i\;\;} & C^n \\ & \searrow \quad {\scriptstyle j}\nearrow & \\ & N & \end{array}$$

(a diagram of C-R maps) commutes, and each map is of maximal rank (i is the natural imbedding). The germ of N at M is called the germ of the minimal flattening of M in C^n, and is unique.

How to obtain N: consider the 'abstract' complexification of M, M_C. M_C is a complex manifold with M a totally real, real analytic submanifold of M_C, and $\dim_C M_C = \dim M$. ("Totally real" means "having no holomorphic tangent vectors.") $T(M)_p \otimes C = T(M_C)_p$. We extend $H(M)$ and $A(M)$ to vector subbundles of $T(M_C)$, perhaps shrinking M_C as a neighborhood of M. Call these bundles H' and A'. Then A' is the conjugate of H', and H' (resp. A') is involutive. Let $\mathfrak{L}'$ be the Lie algebra generated by sections of H' and A'. Then $\mathfrak{L}'$ is a distribution of constant fiber dimension, since $\mathfrak{L}(M)$ is. Let N be the union of all maximal integral submanifolds of $\mathfrak{L}'$ which intersect M. N is the desired 'locally flat' manifold. The map j is induced by $z_1, \ldots, z_n$ on M extended to M_C and restricted to N.

(By a similar method we could also construct the minimal complexification of M in C^n—the smallest germ of a complex submanifold of C^n containing M. Something like this is also done in Tomassini [**3**].)

So M is a subset of a locally holomorphically convex set N (since ex dim $\mathfrak{L}(N)=0$), with dim $N=\dim M+e$. Therefore M is not locally extendible to a set of dimension greater than dim $M+e$.

We can also prove a nongeneric extendibility theorem for differentiable C-R submanifolds with C-R codim $=1$, using a result of Nirenberg and Wells. (Let $\mathfrak{A}(K)$ be the uniformly closed algebra of functions on K generated by restrictions to K of functions in $\mathfrak{H}(K)$.)

THEOREM [2]. *If M is a differentiable hypersurface of C^n, and $p \in M$, then any sufficiently small compact neighborhood K of p in M has the following property: the uniformly closed algebra of functions generated by restriction to K of C-R fucctions on M is identical with $\mathcal{A}(K)$.*

We also need:

LEMMA. *Suppose a compact set K is extendible to a compact set K'. Then every element of $\mathcal{A}(K)$ is the restriction of a unique function in $\mathcal{A}(K')$.*

Then we can get:

THEOREM. *Let M be a nontrivial C-R submanifold of C^n, with C-R* codim $=1$. *If $e=$* ex dim $\mathcal{L}(M)$ *is* 0, *then M is locally holomorphically convex. If $e=1$, M is locally extendible to a set containing a submanifold N, and* dim $N=$ dim $M+1$. *M is not locally extendible to a set of dimension greater than* dim $M+1$. (*Of course, e is* 0 *or* 1.)

PROOF. Suppose dim $M=k$. Then, as before, we can find $m=\frac{1}{2}(k+1)$ complex-valued *C-R* functions $S_1, \ldots, S_m$ so that $S=(S_1, \ldots, S_m)$ is a *C-R* imbedding of M as a hypersurface of C^m near some point $p \in M$. If $e>0$, then $S(M)$ is extendible to a set L^* containing a $(k+1)$ dimensional manifold. To transport L^* back to C^n, proceed as in the real analytic case, but use the preceding theorem and lemma instead of the complexification argument.

The function $z_j|_M (1 \leqq j \leqq n)$ is a *C-R* function; by restricting to a suitable compact neighborhood of p, we can find extensions z_j^* of z_j (considered on $S(M)$) to some compact subset of L^* containing a C^m-open set. These values of z_j^* furnish the desired subset L of C^n, and the extension of any function to L is obtained in the obvious way.

Since we see that functions in $\mathcal{H}(M)$ are locally approximable on M (in small enough compact sets) by analytic functions in C^m, M is not locally extendible to a set of dimension greater than m.

If $e=0$, then the theorem is given in [1].

Some points worthy of further investigation should be noted. Can the theorem of Nirenberg and Wells (closure of *C-R* functions $=\mathcal{A}(K)$) be generalized to higher codimensional submanifolds? Then we could prove theorems similar to the above in higher codimension. If a submanifold is locally holomorphically convex (resp. extendible to a set of dimension at most k) is it holomorphically convex (resp. extendible to a set of dimension at most k)? This is a higher codimensional analogue of the Levi problem, and seems to be difficult.

References

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