A NOTE ON THE IDEAL STRUCTURE OF $C(X)$

WILLIAM E. DIETRICH, JR.

Let F denote the reals or the complexes. For a topological space X, let $C(X)$ stand for the F-algebra of continuous maps $X \rightarrow F$. For $A \subset X$, let $I_A(F_A)$ denote the ideal of maps vanishing on A (on a neighborhood of A). It is well known [5, p. 329] that if X is compact, I is an ideal of $C(X)$, and $A = \bigcap \{Z(f) = f^{-1}(0) \mid f \in I\}$, then $F_A \subset I \subset I_A$. This means that for compact X, an examination of the ideal structure of $C(X)$ can proceed piecewise: it is enough to obtain information about the ideals of the algebras I_A/F_A, $A \subset X$. For arbitrary X, such an approach will not tell all,[1] but it does serve to reveal the algebraic complexity of $C(X)$. For we will show that unless it is trivial, I_A/F_A fails to contain minimal or countably generated maximal ideals, and fails to have minimum or maximum condition even on ideals of the form J/F_A, J an ideal of $C(X)$ lying between F_A and I_A. Corollary 2 is a partial generalization of [2, Theorem 5.2, p. 663] which we apply to show that the countably generated ideals of $C(X)$ are rarely closed in the locally m-convex topology of uniform convergence on the compact parts of X (Corollary 3).

LEMMA. *If $g \in C(X) - F_A$, $g \notin gI_A + F_A$.*

PROOF. Suppose instead that $g = gf + h$ for some $f \in I_A$, $h \in F_A$. h vanishes on some neighborhood U containing A, and since $V = f^{-1}(B(0, 1/2)) \cap U$ is a neighborhood of A and $g \notin F_A$, $g(x) \neq 0$ for some $x \in V$. But then $f(x) = g(x)/g(x) = 1$, a contradiction.

This means in particular that neither I_A/F_A nor I_A has a (nonzero) identity. For choose $g \in I_A - F_A$. If either has an identity, there is an $f \in I_A$ with $fg - g \in F_A$, and hence $g \in gI_A + F_A$ in violation of the lemma. So a distinction should be made between the ring and the algebra ideals of these algebras. "Ideal" will mean ring ideal. Of course if J is an ideal of $C(X)$ lying between F_A and I_A, it is an algebra ideal of I_A and J/F_A is an algebra ideal of I_A/F_A.

THEOREM 1. *I_A/F_A has no (nonzero) minimal (ring or algebra) ideals. In fact if I is a nonzero ideal of I_A/F_A, there is an ideal J of $C(X)$ lying between F_A and I_A such that $0 \subsetneq J/F_A \subsetneq I$.*

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[1] In fact, a completely regular space X is compact iff every proper ideal of $C(X)$ is contained in some I_p, $p \in X$ [3, pp. 58–59].

PROOF. Let $p: I_A \to I_A/F_A$ be the projection map and I a nonzero ideal of I_A/F_A. $p^{-1}(I)$ is an ideal of I_A properly containing F_A, and we may choose an $f \in p^{-1}(I) - F_A$. For each positive integer n, set $J_n = f^n I_A + F_A$. J_n is an ideal of $C(X)$ and $F_A \subset J_{n+1} \subset J_n \subset p^{-1}(I)$. So it is sufficient to show $J_{n+1} \neq J_n$ for each n. If instead $J_{n+1} = J_n$ for some n, $f^{n+1} = f^n f \in J_n = J_{n+1} = f^{n+1} I_A + F_A$. But this contradicts the lemma since $Z(f^{n+1}) = Z(f)$ implies $f^{n+1} \notin F_A$.

COROLLARY 1. *I_A/F_A has minimum condition on ideals of the form J/F_A, J an ideal lying between F_A and I_A iff $I_A/F_A = 0$.*

We can dualize Corollary 1 with "maximum" in place of "minimum," although it is unclear whether the same is true for Theorem 1 (see Theorem 4).

THEOREM 2. *I_A/F_A has maximum condition on ideals of the form J/F_A, J an ideal of $C(X)$ lying between F_A and I_A iff $I_A/F_A = 0$.*

PROOF. If $I_A/F_A \neq 0$, we may choose some $g \in I_A - F_A$. One of Reg, Img, which we denote by f, does not lie in F_A. f is real-valued, so for each positive rational r with odd denominator in lowest terms, $f^r(x) = f(x)^r$ defines an element $f^r \in C(X)$ and we have $Z(f^r) = Z(f)$. Set $I_n = f^{1/3^n} I_A + F_A$, $n = 1, 2, \cdots$. $f^{1/3^n} = f^{1/3^{n+1}} f^{2/3^{n+1}}$, which is a member of I_{n+1} since $Z(f^{2/3^{n+1}}) = Z(f) \supset A$. So we have $F_A \subset I_n \subset I_{n+1} \subset I_A$. Since I_n is an ideal of $C(X)$, it is sufficient to show $I_n \neq I_{n+1}$ for each n. If instead $I_n = I_{n+1}$ for some n, $f^{1/3^n} \in I_{n+1} = I_n = f^{1/3^n} I_A + F_A$, which contradicts the lemma since $Z(f^{1/3^n}) = Z(f)$ and $f \notin F_A$.

We should remark that if X is connected and A is a proper nonempty zero set in X, then $I_A/F_A \neq 0$. For if $A = Z(f)$ for some $f \in C(X)$, f cannot vanish on any neighborhood of A without making $X - A$, A a disconnection of X. Since every closed set in a metric space is a $^G\delta$ (and hence a zero set), this means if X is connected and metrizable and A is closed, I_A/F_A vanishes only in the trivial cases $A = X$ or $\varnothing$.

We now turn to a dual version of Theorem 1. For $B \subset C(X)$, set $\mathrm{Re}B = \{\mathrm{Re}f \mid f \in B\}$ and $\mathrm{Im}B = \{\mathrm{Im}f \mid f \in B\}$ (of course if F is the reals R, $\mathrm{Im}B = 0$). For an ideal I of $C(X)$ set $BI = \{\sum_i f_i g_i \mid f_i \in B,\ g_i \in I$ with sums finite$\}$. Then BI is an ideal of $C(X)$ contained in I. Finally, if $\mathrm{Re}B = B$ and r is a positive rational with odd denominator in lowest terms, let $B^r = \{f^r \mid f \in B\}$.

THEOREM 3. *Let B be a countable subset of I_A. If I_A/F_A is not trivial, there is some proper prime algebra ideal of I_A containing $BI_A + F_A$.*

PROOF. Define $l:R\to R$ by

$$\begin{aligned} l(y) &= 1/\log 2, & y &\geqq \tfrac{1}{2} \\ &= 1/\log 1/y, & 0 &< y < \tfrac{1}{2} \\ &= 0, & y &\leqq 0. \end{aligned}$$

Then $l\in C(R)$ and $l(0)=0$. By L'Hopital's Rule, for each positive integer m, $l(y)^m/y\uparrow\infty$ as $y\downarrow 0$. Since $I_A/F_A\neq 0$, we may assume $\emptyset\subsetneqq B\subset I_A-F_A$. Let $\{f_n \mid n=1, 2, \cdots\}$ be an enumeration of B (with possible repetitions). Define

$$g(x) = \sum_{i=1}^{\infty} 1/2^i(|f_i(x)| \wedge 1).$$

Then $g\in I_A$ and $Z(g)=\bigcap_{i=1}^{\infty} Z(f_i)$. Since each f_i is not in F_A, $g\notin F_A$. Set $h=l\circ g\in I_A$. We claim $h^m\notin BI_A+F_A$ for each positive integer m. If instead $h^m\in BI_A+F_A$ for some m, there are $g_i\in I_A$, $1\leqq i\leqq n$, $k\in F_A$ such that $h^m=\sum_{i=1}^n f_ig_i+k$. Choose an $\epsilon>0$ so small that for $0<y<\epsilon$, $l(y)^m/y\geqq 1$. Find a neighborhood V of A on which k vanishes. Then

$$U = \bigcap_{i=1}^{n} [g_i^{-1}(B(0, 1/n2^i)) \cap f_i^{-1}(B(0, 1))] \cap V \cap g^{-1}(B(0, \epsilon))$$

is a neighborhood of A, and since $g\notin F_A$, there is some $x\in U$ for which $g(x)\neq 0$. For each $1\leqq j\leqq n$, $|f_j(x)|<1$ and so for each $1\leqq i\leqq n$,

$$1/2^i|f_i(x)| \leqq \sum_{j=1}^{n} 1/2^j|f_j(x)| \leqq \sum_{j=1}^{\infty} 1/2^j|f_j(x)| \wedge 1 = g(x).$$

But $0<g(x)<\epsilon$ and $|g_i(x)|<1/n2^i$, $1\leqq i\leqq n$, so we have

$$\begin{aligned} 1 \leqq l(g(x))^m/g(x) &= h^m(x)/g(x) = \left|\sum_{i=1}^{n} f_i(x)g_i(x)/g(x)\right| \\ &\leqq \sum_{i=1}^{n} |g_i(x)|\,|f_i(x)|/g(x) \leqq \sum_{i=1}^{n} 2^i|g_i(x)| < 1. \end{aligned}$$

This contradiction shows $h^m\in I_A-BI_A+F_A$ for each m. Since BI_A+F_A is an algebra ideal of I_A, we can use Zorn's Lemma (as in [**4**, p. 65]) to find a prime algebra ideal of I_A containing BI_A+F_A which does not meet the multiplicatively closed set $\{h^m \mid m=1, 2, \cdots\}$.

COROLLARY 2. *Suppose X is completely regular and A is closed in X. If I_A is countably generated as an ideal of $C(X)$, then A is open.*

PROOF. If I_A is generated as an ideal of $C(X)$ by a countable set D, then $D \subset I_A$ and $I_A = DC(X) = \mathrm{Re}D\ C(X) + \mathrm{Im}D\ C(X) = \mathrm{Re}D^{1/3} I_A + \mathrm{Im}D^{1/3} I_A$ (since for $f \in D$, $\mathrm{Re}f = \mathrm{Re}f^{1/3}\ \mathrm{Re}f^{2/3} \in \mathrm{Re}D^{1/3} I_A$, etc.) $= [\mathrm{Re}D^{1/3} \cup \mathrm{Im}D^{1/3}]\ I_A$. So if we set $B = \mathrm{Re}D^{1/3} \cup \mathrm{Im}D^{1/3}$, B is a countable subset of I_A and $I_A = BI_A + F_A$. However, if $\{f_n \mid n = 1, 2, \cdots\}$ is an enumeration of D we have by the hypothesis on X and A that $A = \bigcap \{Z(f) \mid f \in I_A\} = \bigcap \{Z(f) \mid f \in D\} = Z(\sum_{i=1} 1/2^i |f_i| \wedge 1)$. So A must be open, since otherwise, $I_A / F_A \neq 0$, and we would contradict Theorem 3.

COROLLARY 3. *Let X be a connected completely regular space. A countably generated ideal of $C(X)$ is closed in $C(X)$ with respect to the topology of uniform convergence on compacta iff it is trivial or all of $C(X)$.*

PROOF. Let I be a countably generated ideal of $C(X)$ and set $A = \bigcap \{Z(f) \mid f \in I\}$. Since X is completely regular, the closure of I in $C(X)$ is I_A [1, Theorem 1(i)]. Thus if I is closed, $I = I_A$ and A is open by Corollary 2. But since X is connected, $A = X$ or $\varnothing$, and so $I = I_X$ or $I_\varnothing$; that is, $I = 0$ or $C(X)$. The converse is obvious.

COROLLARY 4. *Suppose X is compact and D is a countable subset of $C(X)$. Then the ideal of $C(X)$ generated by D is sup norm closed in $C(X)$ iff $\bigcap \{Z(f) \mid f \in D\}$ is open in X.*

PROOF. Let I be the ideal generated by D and set $A = \bigcap \{Z(f) \mid f \in D\}$. Since $A = \bigcap \{Z(f) \mid f \in I\}$ and X is compact, $F_A \subset I \subset I_A$. If A is open, $F_A = I_A$, and so $I = I_A$ and is hence closed. Since F_A is sup norm dense in I_A, if I is closed, $I = I_A$, and the result follows from Corollary 2.

Finally, we have the following partial dual of Theorem 1:

THEOREM 4. *I_A / F_A contains no (proper) countably generated maximal (ring or algebra) ideals. In fact, if I is a nonzero countably generated ideal of I_A / F_A, there is an ideal J of $C(X)$ lying between F_A and I_A and a prime algebra ideal P of I_A containing F_A such that $I \subsetneq J/F_A \subset P/F_A \subsetneq I_A/F_A$.*

PROOF. The first assertion follows from the second since if M is a (proper) maximal ideal of I_A/F_A, $M \neq 0$. Suppose I is a nonzero ideal of I_A/F_A generated as an ideal of I_A/F_A by a countable set E. Choose a countable subset D of I_A such that $p(D) = E$ where $p: I_A \to I_A/F_A$ is the projection map. Then $E \subset DC(X)/F_A$, so $I \subset DC(X)/F_A$ and hence $p^{-1}(I) \subset F_A + DC(X) = F_A + SI_A$ where S is the countable set $\mathrm{Re}D^{1/3} \cup \mathrm{Im}D^{1/3}$. $I \neq 0$ means $I_A/F_A \neq 0$ and Theorem 3 then implies the existence of a real-valued $f \in I_A - SI_A + F_A$ (otherwise

$I_A = SI_A + F_A$). Set $B = S \cup \{f^{1/3}\}$ and $J = BI_A + F_A$. Then J is an ideal of $C(X)$ lying between F_A and I_A, and because $f = f^{1/3}f^{2/3} \in J - SI_A + F_A$ and $p^{-1}(I) \subset SI_A + F_A$, J properly contains $p^{-1}(I)$. Applying Theorem 3 to J gives a proper prime algebra ideal P of I_A containing J, and passing to the quotient we have $I \subsetneq J/F_A \subset P/F_A \subsetneq I_A/F_A$ as required.

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Northwestern University