A NOTE ON THE REPRESENTATION OF A SOLUTION OF AN ELLIPTIC DIFFERENTIAL EQUATION NEAR AN ISOLATED SINGULARITY[1]

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There are a number of results known which state that a solution u of an elliptic differential equation

$$Au = 0 \tag{1}$$

which has an isolated singularity at a point $p \in \boldsymbol{R}^n$ may be expressed as the sum of a derivative of the fundamental solution of A and a solution of (1) regular at p, providing that u satisfies one of various conditions limiting its growth near p (see for example F. John [2] or R. Seeley [7]). The main conclusion of this note is a representation of any solution of (1) with an isolated singularity at p which makes no assumption on the behavior of u near the singularity; the representation is in terms of a (real) analytic functional supported on $\{p\}$ applied to the fundamental solution. This result is in the spirit of the work of J. L. Lions and E. Magenes [3] on elliptic boundary value problems with analytic functionals as data.

Actually with our method it involves no additional difficulty to obtain the representation when u is singular on a compact set $K \subset \boldsymbol{R}^n$—that is, when u is a solution of (1) on $\Omega \sim K$, where Ω is some open connected neighborhood of K in $\boldsymbol{R}^n$. We may suppose without loss of generality that $\partial\Omega$ is smooth and that u is $\mathcal{C}^\infty$ on $\bar{\Omega} \sim K$, because any neighborhood of K contains a smaller neighborhood for which this will be true. We assume that A is a properly elliptic differential operator (as defined by M. Schechter in [6]) of order $2m$ whose coefficients are analytic on $\bar{\Omega}$. Let γ be a two-sided fundamental solution for A on Ω; more explicitly, if Γ: $\mathfrak{D}(\Omega) \to \mathcal{E}(\Omega)$ is defined by

$$\Gamma\phi(x) = \int_\Omega dx' \gamma(x, x')\phi(x'),$$

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then

$$\Gamma A\phi = A\Gamma\phi = \phi \tag{2}$$

for all $\phi \in \mathfrak{D}(\Omega)$. The existence of such a fundamental solution was proved by B. Malgrange in [**4**].

To specify the notation we review in this paragraph the terminology of analytic functionals (see A. Martineau [**5**] for details). We define the space of functions $\mathfrak{A}(V)$ analytic on an open set V in $\boldsymbol{R}^n$ as the inductive limit as $\epsilon \to 0$ of the space of functions on V whose power series about any point converges in a ball of radius ϵ. That is, if $\epsilon > 0$, let

$$\|\psi\|_\epsilon = \sup_{x \in V} \sup_{\alpha} (\alpha!)^{-1} \epsilon^{|\alpha|} \left| D^\alpha \phi(x) \right|,$$

where α is a multi-index exponent for the differentiation operator D; let

$$\mathfrak{A}(V;\epsilon) = \{\psi \in \mathcal{E}(V) \mid \|\psi\|_\epsilon < \infty\}$$

and give it the norm topology; and finally let

$$\mathfrak{A}(V) = \operatorname*{ind\,lim}_{\epsilon \to 0} \mathfrak{A}(V;\epsilon).$$

In the usual way, the space of functions $\mathfrak{A}(K)$ analytic on a closed set K is defined as the inductive limit of the space of functions analytic on some neighborhood V of K as V decreases to K. A (real) analytic functional supported on K is a continuous linear functional on $\mathfrak{A}(K)$, an element of the dual space $\mathfrak{A}'(K)$.

We remark that the fundamental solution for A is analytic, because A has analytic coefficients. Thus if $x \in \Omega \sim K$, then $D_x^\alpha \gamma(x, \cdot) \in \mathfrak{A}(K)$ for any multi-index α, and the difference quotients for these derivatives converge in the topology of $\mathfrak{A}(K)$. In particular,

$$A\gamma(x, \cdot) = 0 \in \mathfrak{A}(K)$$

for $x \in \Omega \sim K$. If $T \in \mathfrak{A}'(K)$, we denote by $T[\gamma]$ the function

$$v(x) = T[\gamma(x, \cdot)] \qquad (x \in \Omega \sim K). \tag{3}$$

It is readily shown by an exchange of limits that $Av = 0$. We state now our main theorem.

THEOREM 1. *If u is a solution of* (1) *on $\Omega \sim K$, then there is an analytic functional T supported on K such that $u - T[\gamma]$ is the restriction to $\Omega \sim K$ of an* (*analytic*) *solution of* (1) *defined on Ω.*

Before we prove the theorem we introduce a space of solutions on K of the adjoint equation and we construct from u a certain linear functional on this space which characterizes the singularity of u on K.[3] Let

$$\mathcal{J}(K) = \{\psi \in \mathcal{A}(K) \mid A^*\psi = 0\},$$

and give it the relative topology. If ϕ is a smooth function on $\Omega \sim K$ such that $A\phi \in L^1(\Omega \sim K)$ and if $\psi \in \mathcal{J}(V)$ [that is, the kernel of A^* in $\mathcal{A}(V)$, where V is some neighborhood of K], let ρ be a $\mathcal{C}^\infty$ function supported in V that is identically one near K and define

$$B[\phi, \psi] = \int_{\Omega \sim K} dx\{\phi A^*(\rho\psi) - \rho\psi A\phi\}. \tag{4}$$

Since $A^*(\rho\psi)$ has compact support, the possibly troublesome first term of the integral in (4) is well defined. The integral is independent of the choice of ρ because the difference of two possible choices is a test function supported in $\Omega \sim K$, permitting an integration by parts. For each V the functional $B[\phi, \cdot]$ is continuous on $\mathcal{J}(V)$, so $B[\phi, \cdot] \in \mathcal{J}'(K)$ by inductive limits. The functional $B[u, \cdot]$ specifies the boundary data of u on ∂K in the sense of equation (6) below.

Suppose w is a smooth function on $\bar{\Omega}$ such that A^*w vanishes in a neighborhood V of K; choose ρ as in (4) and let $\zeta = 1 - \rho$, so that ζ is a $\mathcal{C}^\infty$ function vanishing near K. Consider the integral

$$\int_{\Omega \sim K} dx\{uA^*w - wAu\} = \int_{\Omega \sim K} dx\{uA^*(\rho w) - \rho wAu\} + \int_{\Omega \sim K} dx\{uA^*(\zeta w) - \zeta wAu\}; \tag{5}$$

the first term on the right in (5) is simply $B[u, w]$, while the second reduces to a surface integral by Green's theorem. Hence

$$\int_{\Omega \sim K} dx\{uA^*w - wAu\} = B[u, w] + \sum_{j=0}^{2m-1} \int_{\partial\Omega} d\sigma \left(\frac{\partial}{\partial\nu}\right)^j uD_jw, \tag{6}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative and D_j is a differential operator of order $2m-j-1$ for which $\partial\Omega$ is noncharacteristic.

PROPOSITION 2. *If u is a solution of $Au = 0$ on $\Omega \sim K$, then u is the restriction to $\Omega \sim K$ of a solution on Ω if and only if $B[u, \cdot] = 0$.*

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[3] This procedure is very suggestive of the duality considered by A. Grothendieck in [1] and by others.

PROOF. If u extends to a solution of (1) on Ω, then an integration by parts in (4) checks that $B[u, \cdot]=0$. Conversely, suppose that $B[u, \cdot]=0$; we show that an extension of u to Ω may be obtained as a solution u' of the Dirichlet problem, $Au'=0$ in Ω, whose Dirichlet data on $\partial\Omega$ coincides with that of u. A solution u' exists, for if w is any solution of the adjoint equation $A^*w=0$ with homogeneous data, then by (6)

$$\sum_{j=0}^{m-1}\int_{\partial\Omega} d\sigma\left(\frac{\partial}{\partial\nu}\right)^j u D_j w = 0;$$

that is, the data is orthogonal to any solution of the adjoint equation, so a solution exists according to the Fredholm alternative.

Let N denote the finite-dimensional space of solutions of (1) on Ω with vanishing Dirichlet data. If $f\in\mathfrak{D}(\Omega\sim K)$ is orthogonal to N, choose w so that $A^*w=f$ and w has homogeneous data. Then again by (6)

$$\int_{\Omega\sim K} dx\, u' A^* w = \sum_{j=0}^{m-1}\int_{\partial\Omega} d\sigma\left(\frac{\partial}{\partial\nu}\right)^j u' D_j w = \int_{\Omega\sim K} dx\, u A^* w,$$

thus

$$\int dx(u - u')f = 0.$$

Hence $(u-u')$ is orthogonal to any vector in $N^{\perp}$, so u differs from u' by an element of N which may be added to u' to obtain the desired extension.

PROOF OF THEOREM 1. Suppose u is a solution of (1) on $\Omega\sim K$. For $x\in\Omega\sim K$ we define the function

$$v(x) = B[u, \gamma(x, \cdot)].$$

By the Hahn-Banach theorem $B[u, \cdot]$ may be extended from $\mathfrak{J}(K)$ to a linear functional T on $\mathfrak{A}(K)$, so v is of the form $T[\gamma]$. As we remarked before Theorem 1, $Av=0$; we show below that $B[v, \cdot]=B[u, \cdot]$. Hence by the proposition $u-v$ is the restriction to $\Omega\sim K$ of a solution of (1) on Ω.

If $w\in\mathfrak{J}(K)$,

$$B[v, w] = \int dx\, v(x) A^*[\rho w(x)] = \int dx\, T[\gamma(x, \cdot)] A^*[\rho w(x)]$$

$$= T\left\{\int dx\, \gamma(x, \cdot) A^*[\rho w(x)]\right\},$$

since the fact that $A^*(\rho w)$ has compact support in $\Omega \sim K$ implies that the integral converges in the topology of $\mathfrak{A}(K)$. Thus

$$B[v, w] = T[\Gamma^* A^*(\rho w)], \tag{7}$$

where $\Gamma^*\phi(x) = \int dx'\tilde{\gamma}(x', x)\phi(x')$. It is obvious from (2) that $\Gamma^* A^*\phi = A^*\Gamma^*\phi = \phi$ for all $\phi \in \mathfrak{D}(\Omega)$; moreover, since $\rho \equiv 1$ near K, $\Gamma^* A^*(\rho w) = w$ near K. Therefore from (7), $B[v, w] = T[w] = B[u, w]$, where the final equality follows from the observation that $w \in \mathfrak{g}(K)$. This completes the proof.

We remark in closing that a similar representation for a solution of the inhomogeneous equation $Au = f$ can be proved quite simply with our methods.

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