ON THE CONVOLUTION OF LOGARITHMICALLY CONCAVE SEQUENCES

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In [1], Davenport and Polya have considered the following problem. If $\sum p_r' x^r$ and $\sum q_r' x^r$ are two series with positive coefficients and if

$$(1) \qquad (\sum p_r' x^r)(\sum q_r' x^r) = \sum W_r' x^r$$

then what conditions will ensure that the coefficients W_r' shall be logarithmically convex? We say that W_r' is logarithmically convex if

$$(2) \qquad (W_r')^2 \leqq W_{r-1}' W_{r+1}', \qquad r = 1, 2, 3, \cdots .$$

If

$$(3) \qquad p_r = p_r' / \alpha_r,$$

$$(4) \qquad q_r = q_r' / \beta_r,$$

$$(5) \qquad \alpha_r = \frac{\alpha(\alpha + 1) \cdots (\alpha + r - 1)}{1 \cdot 2 \cdot 3 \cdots r},$$

$$(6) \qquad \beta_r = \frac{\beta(\beta + 1) \cdots (\beta + r - 1)}{1 \cdot 2 \cdot 3 \cdots r},$$

$\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$ and if p_r and q_r are both logarithmically convex then Davenport and Polya have proved in [1], that W_r is logarithmically convex, where

$$W_r = \alpha_0 p_0 \beta_r q_r + \alpha_1 p_1 \beta_{r-1} q_{r-1} + \cdots + \alpha_r p_r \beta_0 q_0.$$

It must be noted that the result of Davenport and Polya is false with the omission of the weights α_r and β_r as defined in (5) and (6) respectively. In this paper we prove a similar result for logarithmically concave sequences.

DEFINITION. A sequence $\{\alpha_r\}$ is said to be logarithmically concave if

$$\alpha_r^2 \geqq \alpha_{r-1} \alpha_{r+1}, \qquad (r = 1, 2, 3, \cdots).$$

Evidently a positive sequence is logarithmically concave if and only if it satisfies the relations

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$$\alpha_1/\alpha_0 \geqq \alpha_2/\alpha_1 \geqq \alpha_3/\alpha_2 \geqq \cdots.$$

THEOREM. *Let* $\{p_r\}$ *and* $\{q_r\}$ *be positive logarithmically concave sequences with* $p_0=q_0=1$. *Then the sequence* $\{W_r\}$ *is also logarthmically concave, where the* W_r *are defined by the product of formal power series*

$$\sum_{r=0}^{\infty} W_r x^r = \left(\sum_{r=0}^{\infty} p_r x^r\right)\left(\sum_{r=0}^{\infty} q_r x^r\right). \tag{7}$$

PROOF. Since $(\sum p_r x^r)(\sum q_r x^r) = \sum W_r x^r$, we have

$$W_r = \sum_{j=0}^{r} p_{r-j} q_j. \tag{8}$$

From (8) substituting the values of W_r we have

$$\begin{aligned} W_r^2 - W_{r-1}W_{r+1} = &\left(\sum_{j=0}^{r} p_{r-j}q_j\right)\left(\sum_{j=0}^{r} p_{r-j}q_j\right) \\ &- \left(\sum_{j=0}^{r-1} p_{r-1-j}q_j\right)\left(\sum_{j=0}^{r+1} p_{r+1-j}q_j\right) \end{aligned}$$

or

$$\begin{aligned} W_r^2 - W_{r-1}W_{r+1} = &\left(\sum_{j=0}^{r-1} p_{r-j}q_j\right)\left(\sum_{\lambda=0}^{r} p_{r-\lambda}q_\lambda\right) \\ &- \left(\sum_{j=0}^{r-1} p_{r-1-j}q_j\right)\left(\sum_{\lambda=0}^{r} p_{r+1-\lambda}q_\lambda\right) \\ &+ q_r \sum_{\lambda=0}^{r} p_{r-\lambda}q_\lambda - q_{r+1}\sum_{\lambda=0}^{r-1} p_{r-1-\lambda}q_\lambda. \end{aligned} \tag{9}$$

Now the right side of (9) can be written as I+II+III where

$$\mathrm{I} = \sum_{j=0}^{r-1}\sum_{\lambda=1}^{r} q_j q_\lambda (p_{r-j}p_{r-\lambda} - p_{r-1-j}p_{r+1-\lambda})$$

$$\mathrm{II} = \sum_{j=0}^{r-1} q_j q_0 (p_{r-j}p_r - p_{r-1-j}p_{r+1})$$

$$\mathrm{III} = q_r \sum_{\lambda=0}^{r} p_{r-\lambda}q_\lambda - q_{r+1}\sum_{\lambda=0}^{r-1} p_{r-1-\lambda}q_\lambda.$$

Now III may be rewritten as

$$p_r q_r + \sum_{\lambda=0}^{r-1} p_\lambda (q_r q_{r-\lambda} - q_{r+1}q_{r-1-\lambda})$$

and the expression in the parenthesis is nonnegative by the concavity hypothesis. Thus $\mathrm{III} \geqq 0$. In the same manner it can be proved that $\mathrm{II} \geqq 0$.

We regard I as a sum of terms arranged in an $r \times r$ matrix $(T_{j\lambda})$, with the unusual but understandable indexing $0 \leqq j \leqq r-1$, $1 \leqq \lambda \leqq r$:

$$T_{j\lambda} = q_j q_\lambda (p_{r-j} p_{r-\lambda} - p_{r-1-j} p_{r+1-\lambda}).$$

The diagonal of this matrix is the set of terms $T_{j,j+1}$, where $0 \leqq j \leqq r-1$, and it is clear that all terms on the diagonal vanish. A simple calculation shows that each pair of terms symmetrically positioned with respect to the diagonal has nonnegative sum. Now $\mathrm{I}+\mathrm{II}+\mathrm{III} \geqq 0$ and the theorem is established.

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Reference

1. H. Davenport and G. Polya, *On the product of two power series*, Canad. J. Math. 1 (1949), 1–5.

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