A CHARACTERIZATION OF TOTALLY REGULAR $[J,\ f(x)]$ TRANSFORMS

DANY LEVIATAN AND LEE LORCH

0. **Introduction.** Our main object is to prove a necessary and sufficient condition for $[J, f(x)]$ summation methods to be totally regular (§1). But we take this opportunity to establish also an inclusion theorem for regular $[J, f(x)]$ means (§2).

A *totally regular* summation method is one which sums a sequence $\{s_n\}$, $n=0, 1, \cdots$, to s, whenever $s_n \to s$, both for s finite and for s infinite. Necessary and sufficient conditions for a triangular method to be totally regular were discovered by W. A. Hurwitz [**5**] who also characterized the totally regular Hausdorff methods $[H, g]$ as those generated by nondecreasing $g(t)$, $0 \leqq t \leqq 1$, with $g(0+)=g(0)=0$ and $g(1)=1$. Conditions for total regularity of general methods of summation are more complicated than those for a triangular method and were provided by H. Hurwitz [**4**]. Here we supply a corresponding result for $[J, f(x)]$ methods.

The $[J, f(x)]$ transform was introduced by Jakimovski [**6**] in the following way. Let $f(x)$ be a function differentiable infinitely often in $(0, \infty)$. With a sequence $\{s_n\}$, $n=0, 1, \cdots$, associate the transform

$$(1) \qquad t(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} f^{(n)}(x) s_n, \qquad x > 0.$$

The sequence $\{s_n\}$ is said to be summed to s by $[J, f(x)]$ if the sum on the right-hand side of (1) exists and if $\lim t(x) = s$, $x \to \infty$. (If s is infinite we allow this sum to exist in the extended real number system.)

Some well-known methods are included in the $[J, f(x)]$ transform. The Borel exponential mean is obtained by taking $f(x) = e^{-x}$, and the Abel scale methods A_γ (see Borwein [**2**]) are generated by $f(x) = (1+x)^{-\gamma-1}$, $\gamma > -1$, with $\gamma = 0$ giving the Abel method. The Borel transform and the Abel scale methods are totally regular.

It was proved by Jakimovski [**6**] that the $[J, f(x)]$ transform is regular if and only if

$$(2) \qquad f(x) = \int_0^\infty e^{-ux} d\alpha(u), \qquad x > 0,$$

Received by the editors April 14, 1969.

where

(3) $\alpha(u) \in BV[0, \infty), \quad \alpha(0+) = \alpha(0) = 0, \qquad \alpha(\infty -) = 1.$

For the Borel transform $\alpha(u)=0$, $0\leqq u<1$; $\alpha(u)=1$, $1\leqq u<\infty$. For the Abel scale

$$\alpha(u) = \frac{1}{\Gamma(\gamma+1)}\int_0^u t^\gamma e^{-t}dt.$$

I. **The characterization.** We shall establish the following characterization of totally regular $[J, f(x)]$ methods.

THEOREM 1. *A regular* $[J, f(x)]$ *transform is totally regular if and only if* $\alpha(u)$ *is nondecreasing.*

PROOF. Sufficiency is obvious. Conversely, suppose that the $[J, f(x)]$ transform is totally regular; then, by H. Hurwitz [**4**, Theorem 6],

(4) $$\lim_{x\to\infty} \sum_{n=0}^{\infty}\left[\left|\frac{x^n}{n!}f^{(n)}(x)\right| - \frac{(-x)^n}{n!}f^{(n)}(x)\right] = 0.$$

Hence for each v, $0<v<\infty$, we have

(5) $$\lim_{x\to\infty} \sum_{n\leqq vx}\left[\left|\frac{x^n}{n!}f^{(n)}(x)\right| - \frac{(-x)^n}{n!}f^{(n)}(x)\right] = 0.$$

Since $[J, f(x)]$ is regular, $f(x)$ has the representation (2) where $\alpha(u)$ satisfies (3). We may assume that $\alpha(u)$ is normalized, that is, $\alpha(u) = \frac{1}{2}[\alpha(u+)+\alpha(u-)]$, $0<u<\infty$. Thus, [**7**, Theorem 7d, p. 295] for every t, $0<t<\infty$,

(6) $$\lim_{x\to\infty} \sum_{n\leqq tx} \frac{(-x)^n}{n!}f^{(n)}(x) = \alpha(t).$$

Define

$$\alpha_x(0) = 0,$$

$$\alpha_x(t) = \sum_{n\leqq tx} \frac{(-x)^n}{n!}f^{(n)}(x), \qquad 0 < t < \infty.$$

Then the functions $\alpha_x(t)$, $0<x<\infty$, are of uniformly bounded variation and, by (6), $\lim \alpha_x(t)=\alpha(t)$, $x\to\infty$, $0\leqq t<\infty$. Hence, for $0<v<\infty$,

$$\begin{aligned}\int_0^v |d\alpha(t)| &\leqq \liminf_{x\to\infty} \int_0^v |d\alpha_x(t)| \\ &= \liminf_{x\to\infty} \sum_{n\leqq vx} \left|\frac{x^n}{n!} f^{(n)}(x)\right| \\ &\leqq \limsup_{x\to\infty} \sum_{n\leqq vx} \left|\frac{x^n}{n!} f^{(n)}(x)\right| \\ &\leqq \limsup_{x\to\infty} \int_0^\infty e^{-ux} \sum_{n\leqq vx} \frac{(ux)^n}{n!} |d\alpha(u)|.\end{aligned}$$

Using again [**7**, Theorem 7d, p. 295] we see that the last expression equals

$$\int_0^v |d\alpha(u)|.$$

Thus,

$$\lim_{x\to\infty} \sum_{n\leqq vx} \left|\frac{x^n}{n!} f^{(n)}(x)\right| = \int_0^v |d\alpha(t)|. \tag{7}$$

Combining (5) and (7), it follows that

$$\int_0^v |d\alpha(u)| - \alpha(v) = 0, \qquad 0 < v < \infty,$$

whence $\alpha(u)$ is nondecreasing. This completes the proof.

2. **An inclusion theorem.** The following result deals with the relative strength of regular $[J, f(x)]$ methods. It is an extension of Theorem 5.3 of [**6**].

THEOREM 2. *Suppose that* $[J, f(x)]$ *is a regular transform and that* $\phi(t)$ *satisfies* (3). *Let*

$$g(x) = \int_0^\infty f(xt)\,d\phi(t), \qquad x > 0.$$

If

$$\sum_{n=0}^\infty \frac{x^n}{n!} |f^{(n)}(x)|\, |s_n|$$

exists and is bounded in every finite interval $0 \leqq x \leqq A$, *then* $[J, f(x)]$ *summability of* $\{s_n\}$ *to a finite s implies* $[J, g(x)]$ *summability of* $\{s_n\}$ *to s. In particular,* $[J, g(x)]$ *is regular. Moreover, if* $\phi(t)$ *is nondecreasing the same implication holds also for s infinite.*

PROOF. It follows from our assumptions that $g(x)$ is differentiable infinitely often, and that

$$g^{(n)}(x) = \int_0^\infty t^n f^{(n)}(xt)\,d\phi(t), \qquad x > 0.$$

Thus

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} g^{(n)}(x) s_n = \int_0^\infty \left[\sum_{n=0}^{\infty} \frac{(-xt)^n}{n!} f^{(n)}(xt) s_n \right] d\phi(t).$$

Now, if s_n is $[J, f(x)]$ summable to s and if the sum in the brackets above is bounded in every finite interval $0 \leqq xt \leqq A$, then it follows exactly as for continuous Hausdorff transformations (cf. [**3**, §11.18]) that s_n is $[J, g(x)]$ summable to s. The rest of the proof is also straightforward.

The following immediate consequence of Theorem 2 is an extension of Theorem 5.4 of [**6**].

COROLLARY 1. *If the Borel transform of* $\{|s_n|\}$ *is bounded in every finite interval, and if* $\{s_n\}$ *is Borel summable to a finite s, then* $\{s_n\}$ *is summable to s by all regular* $[J, f(x)]$ *transformations. If s is infinite, then* $\{s_n\}$ *is summable to s by all totally regular* $[J, f(x)]$ *transformations.*

This implies in turn

COROLLARY 2. *Ordinary convergence is not equivalent to any* $[J, f(x)]$ *method.*

PROOF. The sequence $\{0, 1, 0, 1, \cdots\}$ is bounded and Borel summable, hence summable by all regular $[J, f(x)]$ methods. But it is divergent.

REFERENCES

1. S. K. Basu, *On the total relative strength of the Hölder and Cesàro methods*, Proc. London Math. Soc. (2) **50** (1948–49), 447–462.

2. D. Borwein, *On a scale of Abel-type summability methods*, Proc. Cambridge Philos. Soc. **53** (1957), 318–322.

3. G. H. Hardy, *Divergent series*, Oxford Univ. Press, New York, 1949.

4. H. Hurwitz, Jr., *Total regularity of general transformations*, Bull. Amer. Math. Soc. **46** (1940), 833–837.

5. W. A. Hurwitz, *Some properties of methods of evaluation of divergent sequences*, Proc. London Math. Soc. (2) **26** (1927), 231–268.

6. A. Jakimovski, *The sequence-to-function analogues to Hausdorff transformations*, Bull. Res. Council Israel **8F** (1960), 135–154.

7. D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1946.

UNIVERSITY OF ILLINOIS, URBANA AND
YORK UNIVERSITY, TORONTO