ON THE CONVERGENCE OF A SEQUENCE OF PERRON INTEGRALS[1]

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Introduction. This paper is concerned with the convergence of a sequence of Perron integrals. Oscar Perron [**4**] considered the case of a sequence of uniformly convergent Perron integrable functions, and Bauer [**1**] extended Perron's work to functions defined in a space of n-dimensions. McShane [**3**] relaxed the condition of a uniformly convergent sequence of Perron integrable functions and stated necessary conditions for the limit of a sequence of Perron integrals to be the integral of the limit function. The following theorem is a generalization of the above. Throughout this paper integration is in the Perron sense. The Lebesgue integral is denoted by $(\mathfrak{L})\int$.

THEOREM. H1. *$\{f_n(x)\}$ is a sequence of Perron integrable functions whose domain is $I=\{x \mid a \leqq x \leqq b\}$.*

H2. *$f_n(x) \geqq g(x)$ for each n, a.e. (almost everywhere) on I, where $g(x)$ is Perron integrable on I.*

H3. $\lim_n f_n(x) = f(x)$ *a.e. on I.*

Under hypotheses H1–H3, *$f(x)$ is Perron integrable on I, and* $\lim_n \int_a^x f_n(t)dt = \int_a^x f(t)dt$, *if and only if the sequence of integrals $\{\int_a^x [f_n(t)-g(t)]dt\}$ is EAC (equi-absolutely continuous) on I.*

Preliminary theorems. The following theorems are used in the proof of the theorem above. The reader is referred to Kamke [**2**] or McShane [**3**] for a proof of Theorem 1, Theorem 2, and Theorem 3. Vitali [**5**] gave a proof for Theorem 4.

THEOREM 1. *If each of $f_1(x)$ and $f_2(x)$ is a Perron integrable function on I, and k_1 and k_2 are numbers, then $[k_1f_1(x)+k_2f_2(x)]$ is Perron integrable on I, and*

$$\int_a^x [k_1f_1(t) + k_2f_2(t)]dt = k_1\int_a^x f_1(t)dt + k_2\int_a^x f_2(t)dt.$$

THEOREM 2. *If $f(x)$ is Lebesgue integrable on I, then $f(x)$ is Perron integrable on I and*

Presented to the Society, April 12, 1968; received by the editors September 20, 1968.

[1] This is a part of the author's doctoral dissertation, written at the University of Texas at Austin under the direction of Professor H. J. Ettlinger.

$$\int_a^x f(t)dt = (\mathfrak{L}) \int_a^x f(t)dt.$$

THEOREM 3. *If $f(x)$ is Perron integrable on I, and $f(x) \geqq 0$ a.e. on I, then $f(x)$ is Lebesgue integrable on I.*

THEOREM 4. *If $\{f_n(x)\}$ is a sequence of Lebesgue integrable functions for x on I, $\lim_n f_n(x) = f(x)$ a.e. on I, and $f_n(x) \geqq 0$ a.e. on I, then $f(x)$ is Lebesgue integrable and $\lim_n (\mathfrak{L})\int_a^x f_n(t)dt = (\mathfrak{L})\int_a^x f(t)dt$ if and only if the sequence $\{(\mathfrak{L})\int_a^x f_n(t)dt\}$ is EAC on I.*

PROOF OF THEOREM. (i) *Proof of the Sufficiency.* Since $f_n(x) \geqq g(x)$ a.e. on I, then by Theorem 1, Theorem 2, and Theorem 3 we have for x on I

$$\int_a^x [f_n(t) - g(t)]dt = (\mathfrak{L}) \int_a^x [f_n(t) - g(t)]dt$$

and so the sequence $\{(\mathfrak{L})\int_a^x [f_n(t) - g(t)]dt\}$ is EAC on I. Then by Theorem 4,

$$\lim_n (\mathfrak{L}) \int_a^x [f_n(t) - g(t)]dt = (\mathfrak{L}) \int_a^x [f(t) - g(t)]dt$$

and Theorem 2 yields

$$\lim_n \int_a^x [f_n(t) - g(t)]dt = \int_a^x [f(t) - g(t)]dt \qquad \text{for } x \text{ on } I.$$

Now, since $g(x)$ is a Perron integrable function on I, then $f(x)$ is Perron integrable on I, and

$$\lim_n \int_a^x f_n(t)dt = \int_a^x f(t)dt.$$

(ii) *Proof of the Necessity.* Under hypothesis,

$$\lim_n \int_a^x f_n(t)dt = \int_a^x f(t)dt.$$

Then by Theorem 1,

$$\lim_n \int_a^x [f_n(t) - g(t)]dt = \int_a^x [f(t) - g(t)]dt$$

and by Theorem 3,

$$\lim \int_a^x [f_n(t) - g(t)]dt = \lim_n (\mathcal{L}) \int_a^x [f_n(t) - g(t)]dt$$

or,

$$\lim_n (\mathcal{L}) \int_a^x [f_n(t) - g(t)]dt = (\mathcal{L}) \int_a^x [f(t) - g(t)]dt.$$

Hence by Theorem 4, the sequence $\{(\mathcal{L})\int_a^x [f_n(t) - g(t)]dt\}$ is EAC on I, and Theorem 2, yields the required result.

References

1. H. Bauer, *Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen,* Monatsh. Math. **26** (1915), 153–198.

2. E. Kamke, *Das Lebesgue–Stieltjes–Integral,* Teubner, Leipzig Verlagsgesellschaft, 1956.

3. E. J. McShane, *Integration,* Princeton Univ. Press, Princeton, N. J., 1944.

4. O. Perron, *Über den Integralbegriff,* Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Ablielung A. Abhandlung **16** (1914).

5. G. Vitali, *Sull integrazione per serie,* Rend. Circ. Mat. Palermo **23** (1907), 137–155.

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