COMMUTATIVE RIMS IN CLANS WITH ZERO

EDWARD N. FERGUSON

A semigroup is a Hausdorff space with a continuous, associative multiplication defined on it. Standard references are [**1**] and [**2**]. No notational distinction will be made between the semigroup and the underlying space. If S is a compact semigroup and if $e^2=e\in S$, then the (unique) maximal group containing e will be denoted $H(e)$. The (unique) minimal ideal of S will be denoted $M(S)$. A hormos is defined in [**1**]. For our purposes it is sufficient to know that a hormos is a compact, connected, and abelian semigroup with identity. This paper relies heavily on the Second Fundamental Theorem of [**1**] and we state the pertinent parts:

SECOND FUNDAMENTAL THEOREM. *Let S be a compact semigroup. Then the following properties are equivalent:*

(1) *The connected component of each idempotent meets the minimal ideal.*

(2) *If e is an idempotent, there is an irreducible hormos T in eSe such that $T\cap H(e)=\{e\}$ and $T\cap M(S)\neq\emptyset$.*

(3) *If e is an idempotent and A_e a compact, connected, abelian subgroup of $H(e)$, then there is an irreducible hormos T in the centralizer of A_e and in eSe such that $T\cap H(e)=\{e\}$ and $T\cap M(S)\neq\emptyset$.*

Throughout this paper cohomology is that of Alexander-Spanier with coefficient group arbitrary.

DEFINITION. Let X be a space and R a compact subset of X. Then R is called a *rim* of X if for each proper closed subset Y containing R there is some integer n such that the map $H^n(X)\rightarrow H^n(Y)$ induced by inclusion is not onto. (See [**3**], but note the slight difference.)

LEMMA. *Let S be a continuum semigroup containing* 0 *and* 1 *and let R be a rim of S. If T is a continuum subsemigroup containing* 0 *and* 1, *then $S=RT$.*

PROOF. It is clear that $R\subseteq RT$ and that RT is homotopic to 0. Hence $H^n(S)\rightarrow H^n(RT)$ is onto for all n. Since R is a rim, $RT=S$. This lemma admits a quick generalization to actions.

THEOREM. *Let S be a continuum semigroup with* 0 *and* 1 *and let R be a rim of S. Assume R is connected and if $x, y\in R$, then $xy=yx$. Then S is commutative.*

Presented to the Society, August 28, 1969; received by the editors April 15, 1969.

PROOF. It is known [**3**] that $1\in R$. Let C be the smallest compact semigroup in S containing R. C is a continuum subsemigroup, $1\in C$, and C is abelian. Therefore if $e\in M(C)$, then $M(C)=H(e)\cap C$. Also $eSe\cap C=H(e)\cap C$.

By the SFT there is a hormos I in C from 1 to $H(e)\cap C$. There is also a hormos T in eSe from e to 0 and $T\cap H(e)=\{e\}$. We may choose T in the centralizer of $M(C)$ so that $M(C)T$ is a semigroup. Relying on the lemma we have $R(I\cup M(C)T)=S$. Now $RI\subseteq C$ so $eSe\subseteq RM(C)T$. But $RM(C)T\subseteq M(C)T\subseteq H(e)T\subseteq eSe$. Thus $eSe=RM(C)T=M(C)T$ an abelian semigroup.

Now $S=C\cup eSe$ and both members of the union are abelian. Let $c\in C$ and $d\in eSe$. Then $cd=ced=dce=dece=dec=dc$. Hence members of C and eSe commute with each other and S is abelian.

EXAMPLE. Let $S_1=\{(a,\ b)\mid(a,\ b)\in R_2,\ |a|+|b|\leqq 1\}$. Define a multiplication on S_1 by $(a,\ b)(c,\ d)=(|ac|,\ |ad|+|b|)$. $M(S_1)=\{(0,\ x)\mid 0\leqq x\leqq 1\}$. $S_1/M(S_1)$ (the Rees quotient) is a compact semigroup with 0 and connected, commutative rim. However, $S_1/M(S_1)$ is not commutative.

Let S_2 be $[0,\ 1]$ with the usual multiplication.

Take the disjoint union of $S_1/M(S_1)$ and S_2 and identify their respective zeros. If $(a,\ b)\in S_1/M(S_1)$ and $x\in S_2$, define

$$(a, b)x = (ax, b),$$
$$x(a, b) = (ax, bx).$$

The space S so defined is a continuum semigroup with zero and 1. It has for a rim the rim of $S_1/M(S_1)$ together with the identity. The rim is commutative, but S is not.

The author wishes to express his thanks to Professor D. Stadtlander for many helpful conversations.

BIBLIOGRAPHY

1. Karl Heinrich Hofmann and Paul S. Mostert, *Elements of compact semigroups*, Merrill, Columbus, Ohio, 1966.

2. A. B. Paalman-deMiranda, *Topological semigroups*, Mathematische Centrum, Amsterdam, 1964.

3. A. D. Wallace, *The Gebielstreue in semigroups*, Indag. Math. **18** (1956), 271–274.

UNIVERSITY OF FLORIDA