SYMBOLIC CALCULUS FOR ALGEBRAS OF FOURIER-STIELTJES TRANSFORMS

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The purpose of this note is to prove Theorem 1 below. G is a nondiscrete locally compact abelian group with dual group Γ. The algebra of Fourier transforms on Γ is $A(\Gamma)$, and the algebra of Fourier-Stieltjes transforms is $B(\Gamma)$. If μ is a measure in G, $\hat{\mu}$ is its Fourier-Stieltjes transform, and $\|\hat{\mu}\|$ is the total variation norm $\|\mu\|$.

THEOREM 1. *Let X be a closed subalgebra of $B(\Gamma)$ such that*

(i) $A(\Gamma)\subseteq X$,

(ii) $\hat{\delta}_0\in X$ *($\hat{\delta}_0$ is the identity of $B(\Gamma)$),*

(iii) *X contains a real function which is the transform of a measure whose powers are pairwise mutually singular.*

Suppose F is a complex valued function defined on $[-1, 1]$ and suppose $F\circ f\in B(\Gamma)$ for all $f\in X$ such that $f(\Gamma)\subseteq[-1, 1]$. Then F is the restriction to $[-1, 1]$ of an entire function.

Once Lemma 6.3.6 of [2] is extended to our version (Lemma 2 below), the proof of Theorem 1 follows the pattern of [2, 6.3]. We shall let $F\circ\mu$ be the unique measure having $F\circ\hat{\mu}$ as its transform (when $F\circ\mu$ is defined).

LEMMA 1 [2, 2.6.7]. *Let C be a compact subset of Γ, Γ a locally compact group and $\epsilon>0$. Then there is a Fourier transform $f\in A(\Gamma)$ such that*

(i) $\|f\|\leqq 1+\epsilon$,

(ii) *support f is compact,*

(iii) *f is one on C, and $0\leqq f\leqq 1$.*

LEMMA 2. *Suppose F is defined on all of the line R, and X satisfies the hypotheses of Theorem 1, and $F\circ f\in B(\Gamma)$ whenever f is a real function in X. Then for each real function $\hat{\mu}\in X$ there exist numbers $\eta>0$, K such that $\nu\in X$ a real function and $\|\nu\|<\eta$ imply*

$$\|F\circ(\mu+\nu)\| < K. \tag{1}$$

PROOF. If the conclusion is false, then there exist μ and $\nu_j\in X$ such that $\|\nu_j\|<2^{-j}$, $\hat{\mu}(\Gamma)\subset R$, $\hat{\nu}_j(\Gamma)\subseteq R$, and $\|F\circ(\mu+\nu_j)\|>j$, $j=1, 2, 3, \cdots$. We may, in fact, by applying Lemma 1 and [2, 1.9.1] assume that $\nu_j\in L^1(G)$ and $\hat{\nu}_j$ has compact support. Indeed,

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$\|F\circ(\mu+\nu_j)\|>j$ implies by [2, 1.9.1] that there exists a trigonometric polynomial

$$P(x) = \sum_{k=1}^{N} c_k(x, \gamma_k), \qquad \gamma_k \in \Gamma \tag{2}$$

with

$$\|P\|_\infty = \sup\{\,|P(x)| : x \in G\} \leqq 1 \tag{3}$$

and

$$\left|\sum_k c_k F\circ((\hat\mu + \hat\nu_j)(\gamma_k))\right| > j. \tag{4}$$

Let $q\in L^1(G)$ be chosen by Lemma 1 so that $\|q\|_1<2$ and $\hat q=1$ on support $\hat P(=\{\gamma_k\}_{k=1}^N)$. Then $\|F\circ(\mu+q*\nu_j)\|>j$.

The basic trick now is to "adjust" the ν_j so that the supports of their transforms are "very" disjoint. That way we can add up the ν_j (since $\|\nu_j\|<2^{-j}$ we get an element of $L^1(G)$), $f=\sum\nu_j$, and then pick out $F\circ(\mu+\nu_j)$ from $F\circ(\mu+f)$ by multiplying the latter (roughly speaking) by a function $h_j\in L^1(G)$ such that $\hat h_j=1$ on the support of $\hat\nu_j$, but $\hat h_j\cdot\hat h_k=0$ if $j\neq k$. We then will get an equality of the form

$$h_j * F\circ(\mu+f) + (\delta_0 - h_j) * F\circ(\mu) = F\circ(\mu+\nu_j). \tag{5}$$

Now take norms and apply the triangle inequality and the multiplication inequality. We obtain

$$\|h_j\|\,\|F\circ(\mu+f)\| + \|(\delta_0-h_j)\|\,\|F\circ(\mu)\| \geqq \|F\circ(\mu+\nu_j)\| > j. \tag{6}$$

The h_j will be chosen with $\|h_j\|<6$, so we have the absurdity:

$$6\|F\circ(\mu+f)\| + 7\|F\circ(\mu)\| > j \qquad \text{for each } j. \tag{7}$$

Before we give the details of the adjustments we must make to the ν_j, we first observe that if the supports of the ν_j all lie in the same compact subset of Γ, then $\|F\circ(\mu+\nu_j)\|$ must be bounded. Indeed, let support $\hat\nu_j\subset C$ for all j. Let g, $h\in L^1(G)$ be such that support $\hat g$ and support $\hat h$ are compact, $\|g\|<2$, $\|h\|<2$, $\hat g=1$ on C and $\hat h=1$ on support $\hat g$. Then $F\circ(\mu+\nu_j)=g*F\circ(\mu+\nu_j)+(\delta_0-g)*F\circ(\mu+\nu_j)$ $=g*F\circ(h*\mu+\nu_j)+(\delta_0-g)*F\circ\mu$. We now need only show that the term $F\circ(h*\mu+\nu_j)$ is norm-bounded.

Case I. Γ *discrete.* We may use the arguments of [2, 6.3.4] to show that F is continuous. Let N be the number of elements of $C'=$ support $\hat h$. Then N is finite. Let $L=\sup\{\,|F(z)|:z\in(\hat\mu+\hat\nu_j)(C'), j=1,2,\cdots\}$.

L is also finite since the sup is over a relatively compact set inside domain F. Then

$$\|F \circ (h * \mu + \nu_j)\| \leqq NL. \tag{8}$$

This follows because $h * \mu + \nu_j$, and $F \circ (h * \mu + \nu_j)$ are trigonometric polynomials and G is compact.

Case II. Γ *nondiscrete*. Then by [2, 6.5.1 and 6.6.1], we may assume F is analytic in a neighborhood of R. Now, $h * \mu + \nu_j \in L^1(G)$. The (Cauchy) formula of Gelfand [1, Theorem 19] applies. Thus $F \circ (h * \mu + \nu_j)$ is norm-bounded.

We are now ready to adjust the ν_j. We will obtain functions g_j and $h_j \in L^1(G)$ such that (i) $\|g_j\|_1 < 2^{-j}$, (ii) support $\hat{g}_j \subseteq \{\hat{h}_j = 1\}$, (iii) $0 \leqq \hat{h}_j \leqq 1$, (iv) $\|h_j\|_1 < 6$, (v) support $\hat{h}_j$ is compact, (vi) $\|F \circ (\mu + g_j)\| > j$ and (vii) $h_j * h_k = 0$ if $j \neq k$. We will construct these functions by induction. Let $g_1 = \nu_1$, and pick any $h_1 \in L^1(G)$ satisfying (ii)–(v). One exists by [2, 2.6.7]. Now assume we have found $g_1, \cdots, g_n$ and $h_1, \cdots, h_n$ satisfying (i)–(vii) for $j, k \leqq n$. We construct g_{n+1} and h_{n+1}.

Pick $a, b, c \in L^1(G)$ such that $0 \leqq \hat{a}, \hat{b}, \hat{c}, \leqq 1$, $\|a\|_1, \|b\|_1, \|c\|_1 < 2$, and support $\hat{h}_j \subseteq \{\hat{a} = 1\} \subseteq$ support $a \subseteq \{\hat{b} = 1\} \subseteq$ support $b \subseteq \{\hat{c} = 1\}$, and support $\hat{c}$ is compact. This can be done by repeated application of Lemma 1.

We claim $\|F \circ (\mu + (\delta_0 - a) * \nu_j)\| \to \infty$ as $j \to \infty$. Indeed,

$$\begin{aligned} F \circ (\mu + \nu_j) &= b * F \circ (\mu + \nu_j) + (\delta_0 - b) * F \circ (\mu + \nu_j) \\ &= b * F \circ (c * \mu + c * \nu_j) \\ &\quad + (\delta_0 - b) * F \circ (\mu + (\delta_0 - a) * \nu_j). \end{aligned} \tag{9}$$

The first term on the right-hand side of (9) is norm-bounded since support $\hat{c}\hat{\nu}_j \subseteq$ support $\hat{c}$ (which was chosen compact) for all j. Therefore, $\|F \circ (\mu + (\delta_0 - a) * \nu_j)\|$ is increasing. Pick j so large that $\|F \circ (\mu + (\delta_0 - a) * \nu_j)\| > n + 1$ and such that $\|\nu_j\| < 2^{-(n+3)}$. Let $g_{n+1} = (\delta_0 - a) * \nu_j$. Then support $\hat{g}_{n+1}$ is compact, so there exists $f \in L^1(G)$ such that $0 \leqq \hat{f} \leqq 1$, $\|f\|_1 < 2$, support $\hat{f}$ compact and $\hat{f} = 1$ on support $\hat{g}_{n+1}$. Let $h_{n+1} = (\delta_0 - a) * f$. Then $g_1, \cdots, g_{n+1}$ and $h_1, \cdots, h_{n+1}$ satisfy (i)–(viii) above. This completes the induction.

Now set $g = \sum_{n=1}^{\infty} g_n$. Then

$$\begin{aligned} F \circ (\mu + g_n) &= (\delta_0 - h_n) * F \circ (\mu + g_n) + h_n * F \circ (\mu + g_n) \\ &= (\delta_0 - h_n) * F \circ \mu + h_n * F \circ (\mu + g). \end{aligned} \tag{10}$$

This now yields the absurdity of (7).

Examples of algebras X which satisfy (i)–(iii) are easily given: Let E be a perfect compact independent subset of G. Such a set exists if G is not discrete [2, 5.2.2]. Let X be the closed subalgebra of $B(\Gamma)$ which is generated by $A(\Gamma)$, $\hat{\delta}_0$, and the transforms of continuous measures supported on $E \cup -E$. One need only apply [2, 5.3.2] to show that X satisfies (iii).

Varopoulos [3] has shown that if G is not discrete, then only entire functions operate on $B_0(\Gamma) = \{\hat{\mu}: \hat{\mu} \in B(\Gamma),\ \hat{\mu}$ vanishes at infinity$\}$. $B_0(\Gamma) \cup \{\hat{\delta}_0\}$ is a closed subalgebra of $B(\Gamma)$ which satisfies (i)–(iii). Thus our Theorem 1 yields a variant of Varopoulos' result.

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