PROPERTIES OF VECTOR VALUED FINITELY ADDITIVE SET FUNCTIONS

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Suppose that Σ is an algebra of subsets of a set S. If B is a Banach space over the real numbers R, then $H(B)=H(S, \Sigma, B)$ denotes the space of bounded ($\sup_{E\in\Sigma}\|\beta(E)\|_B<\infty$), B-valued, finitely additive set functions β on Σ. Suppose that B has a basis $\{b_i\}$ of unit vectors. Then the coefficient functionals β_i, $i\geqq 1$, of β, determined by $\beta(E)=\sum b_i\beta_i(E)$, are elements of $H(R)$. The purpose of this paper is to initiate a study of the interplay between β, the sequence $\{\beta_i\}$, and the basis $\{b_i\}$. Properties which are obtained are used to establish some Radon-Nikodým theorems relating $\beta\in H(B)$ and $\alpha\in H(A)$, A a Banach space over R.

Let us begin by recalling that because $\{b_i\}$ is a basis for B and $\|b_i\|=1$, $i\geqq 1$, it follows [**8**, p. 67, Theorem 1] that there exists $M_B>0$ such that if $\sum b_i\lambda_i\in B$, then $\|\sum_{i\leqq n} b_i\lambda_i\|\leqq M_B\|\sum b_i\lambda_i\|$ and, hence, $\|\sum_{m<i\leqq n} b_i\lambda_i\|\leqq 2M_B\|\sum b_i\lambda_i\|$. In particular, $|\lambda_i|\leqq 2M_B\|\sum b_i\lambda_i\|$, $i\geqq 1$. Thus the coefficient functionals β_i of β are bounded by $2M_B|\beta|_s(S)$ where $|\beta|_s(S)=\sup\{\|\beta(E)\|;\ E\in\Sigma,\ E\subset S\}$. Since the functionals β_i are also finitely additive, $\{\beta_i\}$ is a bounded sequence of elements of $H(R)$. Moreover, since $\sum b_i\lambda_i\in B$ implies that $\lim \lambda_i=0$, it follows that $\lim \beta_i(E)=0$, $E\in\Sigma$. This latter property permits us to assert that the sequence $\{\beta_i\}$ converges weakly to zero if Σ is a sigma algebra [**6**]. The following example shows that, in general, this is not the case.

EXAMPLE 1. Let $e_i=\{\delta_{ij}\}_{j\geqq 1}$, $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$. Then $\{e_i\}$ is a sequence of unit vectors which is a basis for the space c_0 of sequences of real numbers which converge to zero. Let Σ_0 be an algebra of subsets of a set S_0 such that there exists a bounded sequence $\{\gamma_i\}$ of elements of $H(S_0, \Sigma_0, R)$ satisfying

(1) $\lim_i \gamma_i(E)=0$, $E\in\Sigma_0$ and

(2) the sequence $\{\gamma_i\}$ does not converge weakly to zero.

Let $\gamma\in H(S_0, \Sigma_0, c_0)$ be defined by $\gamma(E)=\sum e_i\gamma_i(E)$.

Example 1 shows that if we wish to conclude that $\{\beta_i\}$ converges weakly to zero, then we must impose additional conditions somewhere. We shall first consider a condition on β and then conditions on B. The condition which we shall impose on β is a natural condition

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to impose when seeking to compare β with an element α of $H(A)$ (see [4, particularly the concluding remarks]). The conditions, G_1 and G_2, on B are technical conveniences which, fortunately, are of sufficiently common occurrence to warrant their introduction.

Before proceeding to Theorem 1, let us recall that if $\mu \in H(R)$ and $E \in \Sigma$, then $v(\mu, E) = \sup\{\mu(F); F \in \Sigma, F \subset E\} - \inf\{\mu(F); F \in \Sigma, F \subset E\}$ is the variation of μ on E.

THEOREM 1. *Suppose that* $\lim \|\beta(F_j)\| = 0$ *whenever* $\{F_j\}$ *is a sequence of pairwise disjoint elements of* Σ. *Then the sequence* $\{\beta_i\}$ *converges weakly to zero.*

PROOF. Suppose, on the contrary, that $\{\beta_i\}$ does not converge weakly to zero. Then, by Theorem 3.1 of [11] (cf. [5]), there exists $\epsilon > 0$ and a sequence $\{E_p\}$ of pairwise disjoint elements of Σ such that $|\sum_p \beta_i(E_p)| > 3\epsilon$ for infinitely many positive integers i. Let i_1 be the least such positive integer, and let p_1 be the least positive integer satisfying $|\sum_{p \leq p_1} \beta_{i_1}(E_p)| > 3\epsilon$. Let $F_1 = \bigcup_{p \leq p_1} E_p$. The following facet of this construction is not necessary to establish Theorem 1 but will be convenient for our proof of Theorem 2. Let q_1 be the least positive integer greater than p_1 for which $\sum_{p \geq q_1} v(\beta_{i_1}, E_p) < \epsilon$. Let k_1 be the least positive integer such that $i \geq k_1$ implies that $\sum_{p<q_1} |\beta_i(E_p)| < \epsilon$. This completes the first stage of this construction; to set the pattern, the second stage follows. Let i_2 be the least positive integer greater than k_1 ($k_1 \geq i_1$) such that $|\sum_p \beta_{i_2}(E_p)| > 3\epsilon$, and let p_2 be the least positive integer satisfying $|\sum_{p \leq p_2} \beta_{i_2}(E_p)| > 3\epsilon$. Let $F_2 = \bigcup_{q_1 \leq p \leq p_2} E_p$. Then $|\beta_{i_2}(F_2)| > |\sum_{p \leq p_2} \beta_{i_2}(E_p)| - \sum_{p<q_1} |\beta_{i_2}(E_p)| > 2\epsilon$. Let q_2 be the least positive integer greater than p_2 such that $\sum_{p \geq q_2} v(\beta_{i_2}, E_p) < \epsilon$. Let k_2 be the least positive integer such that $i > k_2$ implies that $\sum_{p<q_2} |\beta_i(E_p)| < \epsilon$. Repeating this process inductively, we obtain a sequence $\{F_i\}$ of pairwise disjoint elements of Σ and an increasing sequence $i_1, i_2, \cdots$ of positive integers such that

(1) $\beta_{i_j}(F_j) > 2\epsilon$ and

(2) if $k \geq j$, then

$$|\beta_{i_j}(\bigcup_{n \leq k} F_n)| \geq - \sum_{p<q_{k-1}} |\beta_{i_j}(E_p)| + |\sum_{q_{j-1} \leq p \leq p_j} \beta_{i_j}(E_p)| - \sum_{p \geq q_j} v(\beta_{i_j}, E_p) > \epsilon.$$

Notice that up to now we have not used the hypothesis of Theorem 1. Invoking now the hypothesis $\|\beta(F_j)\| \to 0$, a contradiction obtains from (1) and $|\beta_{i_j}(F_j)| \leq 2M_B \|\beta(F_j)\|$.

The following example shows that the converse to Theorem 1 is not true.

EXAMPLE 2. Suppose that N is the set of positive integers and that Γ is the sigma algebra of all subsets of N. Suppose that L is the linear span, over R, of the sequences $e_i = \{\delta_{ij}\}_{j\geqq 1}$, is normed by

$$\|\{x\}_i\| = \sup\{|x_1|, (|x_2| + |x_3|), (|x_4| + |x_5| + |x_6|), \cdots\},$$

and let X denote the completion of L. Then X is represented as a space of sequences. Moreover, because $\lim x_i = 0$, whenever $x = \{x_i\} \in X$, it follows that $\{e_i\}$ is a basis for X. For $i \geqq 1$, let $\beta_i \in H(N, \Gamma, R)$ be defined by $\beta_i(E) = 1/k$ if $i \in E_k = \{i: (k-1)k < 2i \leqq k(k+1)\}$ and $i \in E$, and $\beta_i(E) = 0$ otherwise. Let $\beta(E) = \sum e_i\beta_i(E)$. Then $\beta \in H(N, \Gamma, X)$ and, hence because Γ is a sigma algebra, $\{\beta_i\}$ converges weakly to zero. Moreover, $\lim v(\beta_i, S) = 0$ which of course, also implies weak convergence to zero. Nevertheless, $\{E_k\}$ is a sequence of pairwise disjoint elements of Γ and $\|\beta(E_k)\| = 1$, $k \geqq 1$.

If $\{\beta_i\}$ is a sequence of elements of $H(R)$ satisfying $\Sigma v(\beta_i, S) < \infty$ and β is defined by $\beta(E) = \sum b_i\beta_i(E)$, then $\beta \in H(B)$ and $\lim \|\beta(E_i)\| = 0$ whenever $\{E_i\}$ is a sequence of pairwise disjoint elements of Σ. (Even more is true, namely, $\Sigma\|\beta(E_i)\| < \infty$.) Thus additional conditions on B are not necessary in order that the hypothesis of Theorem 1 be satisfied. We shall introduce two growth conditions on a Banach space with basis and show (Theorem 3) that one of them implies the hypothesis of Theorem 1.

The statement that $\{b_i\}$ satisfies condition G_1 means that if each of M and ϵ is a positive number, then there is a positive number $\delta = \delta(M, \epsilon)$ such that if n is a positive integer, $x = \sum_{i\leqq n} b_i\lambda_i$, $y = b_{n+1}\lambda_{n+1}$, $\|x\| \leqq M$, and $\|y\| \geqq \epsilon$, then $\|x+y\| \geqq \|x\| + \delta$.

Notice that if condition G_1 is satisfied, then $\{b_i\}$ is a monotone basis: $\|\sum_{i\leqq n} b_i\lambda_i\| \leqq \|\sum_{i\leqq n+1} b_i\lambda_i\|$. We might also mention that $\delta(M, \epsilon) = M\delta(1, \epsilon/M)$.

The statement that $\{b_i\}$ satisfies condition G_2 means that if (M, ϵ) is a pair of positive numbers, then there is a positive number $\gamma(M, \epsilon)$ such that if $1 \leqq n < p$, $x = \sum_{i\leqq n} b_i\lambda_i$, $y = \sum_{n<i\leqq p} b_i\lambda_i$, $\|x\| \leqq M$, and $\|y\| \geqq \epsilon$, then $\|x+y\| \geqq \|x\| + \gamma(M, \epsilon)$.

It is clear that if $\{b_i\}$ satisfies G_2, then it also satisfies G_1 ($\delta(M, \epsilon) \geqq \gamma(M, \epsilon)$). We are unaware of an example where G_1 is satisfied but G_2 is not. We shall next show that if G_1 is satisfied, then $\{\beta_i\}$ converges weakly to zero. Then we shall show that if G_2 is satisfied, then the hypothesis of Theorem 1 is satisfied.

THEOREM 2. *Suppose that $\{b_i\}$ satisfies condition G_1. Then the coefficient functionals β_i converge weakly to zero.*

PROOF. Suppose again that $\{\beta_i\}$ does not converge weakly to zero. Then all but the last sentence of the proof of Theorem 1 applies. Let $M = M_B|\beta|_s(S)$. Let k be a positive integer satisfying $k\delta > M$, and let $E = \bigcup_{j\leqq k} F_j$. Then the contradiction $M < \delta k < \sum_{j\leqq k}(\|\sum_{n\leqq i_j} b_n\beta_n(E)\| - \|\sum_{n<i_j} b_n\beta_n(E)\|) \leqq \|\sum_{n\leqq i_k} b_n\beta_n(E)\| \leqq M_B\|\beta(E)\| \leqq M_B|\beta|_s(S)$ follows from (2) of the proof of Theorem 1 together with the hypothesis of Theorem 2.

THEOREM 3. *Suppose that* $\{b_i\}$ *satisfies condition* G_2. *Then* $\lim_i\|\beta(E_i)\| = 0$ *whenever* $\{E_i\}$ *is a sequence of pairwise disjoint elements of* Σ.

PROOF. Suppose on the contrary that there exists $\epsilon > 0$ and a sequence $\{E_n\}$ of pairwise disjoint element of Σ such that $\|\beta(E_n)\| > 3\epsilon$, $n \geqq 1$. Then there exists a positive integer k_1 such that $\|\sum_{i\leqq k_1} b_i\beta_i(E_1)\| > 3\epsilon$ and, moreover, if $k_1 \leqq p < n$, then $\|\sum_{p<i\leqq n} b_i\beta_i(E_1)\| < \epsilon/2$. Let $n_1 = 1$. There exists a positive integer n_2 such that $\sum_{i\leqq k_1}\sum_{j\geqq n_2} v(\beta_i, E_j) < \epsilon$. There exists a positive integer k_2 such that

$$\Big\|\sum_{i\leqq k_2} b_i\beta_i(E_{n_2})\Big\| > 3\epsilon$$

and, moreover, if $k_2 \leqq p < n$, then $\|\sum_{p<i\leqq n} b_i\beta_i(E_{n_2})\| < \epsilon/2^2$. There exists a positive integer n_3 such that $\sum_{i\leqq k_2}\sum_{j\geqq n_3} v(\beta_i, E_j) < \epsilon$. Iterate this procedure. Let m be a positive integer. Then

$$\Big\|\sum_{i\leqq k_1} b_i\beta_i\Big(\bigcup_{j\leqq m} E_{n_j}\Big)\Big\| \geqq \Big\|\sum_{i\leqq k_1} b_i\beta_i(E_{n_1})\Big\| - \sum_{i\leqq k_1}\sum_{j\geqq n_2} v(\beta_i, E_j) > 2\epsilon.$$

For $p > 1$, let $S_p = \{i;\ k_{p-1} < i \leqq k_p\}$. Then, for $1 < p < m$,

$$\begin{aligned}
&\Big\|\sum_{i\in S_p} b_i\beta_i \bigcup_{j\leqq m}(E_{n_j})\Big\| \\
&\qquad \geqq -\sum_{j<p}\Big\|\sum_{i\in S_p} b_i\beta_i(E_{n_j})\Big\| + \Big\|\sum_{i\in S_p} b_i\beta_i(E_{n_p})\Big\| - \sum_{i\in S_p}\sum_{j\geqq n_{p+1}} v(\beta_i, E_j) \\
&\qquad > -\epsilon(1/2 + 1/2^2 + \cdots + 1/2^{p-1}) + 3\epsilon - \epsilon > \epsilon.
\end{aligned}$$

For $p = m$, the preceding inequalities are easily modified to yield that $\|\sum_{i\in S_p} b_i\beta_i\bigcup_{j\leqq m}(E_{n_j})\| > -\epsilon + 3\epsilon = 2\epsilon$. Let m be a positive integer satisfying $m\gamma(|\beta|_s(S), \epsilon) > |\beta|_s(S)$, and let $H = \bigcup_{j\leqq m} E_{n_j}$. Then

$$\begin{aligned}
\|\beta(H)\| &\geqq \Big\|\sum_{i\leqq k_1} b_i\beta_i(H)\Big\| + \sum_{j\leqq m}\Big(\Big\|\sum_{i\leqq k_j} b_i\beta_i(H)\Big\| - \Big\|\sum_{i\leqq k_{j-1}} b_i\beta_i(H)\Big\|\Big) \\
&\geqq m\gamma(|\beta|_s(S), \epsilon) > |\beta|_s(S)
\end{aligned}$$

which is impossible.

Let us now turn to a discussion of some Radon-Nikodým theorems. We begin with brief resumé of some pertinent history.

The classical Radon-Nikodým theorem (e.g. [9, Theorem III.10.2]) asserts that if Σ is a sigma algebra and λ is a countably additive element of $H(C)$ where C denotes the complex numbers, then λ can be given an integral representation with respect to a nonnegative, countably additive element μ of $H(R)$ if, and only if, λ is absolutely continuous with respect to μ.

In 1939, S. Bochner published a generalization [1] which removed the restrictions that Σ be a sigma algebra and that the set functions be countably additive. A representation for the case where $\mu \in H(R)$ appeared [2] in 1962. Theorem III.10.7 of [9] supplements the classical theorem by allowing μ to be complex valued, and recently C. Fefferman [10] extended this latter result to the case of a general algebra of subsets of a set. Thereafter, E. Green and the author [7] gave a proof of Fefferman's result, based on [2]. The approach used in [7] will be applied hereinafter to elements of $H(B)$.

While there are several possible definitions of absolute continuity which might be appropriate, we choose the following.

The statement that $\alpha \in H(A)$, A a Banach space, is absolutely continuous with respect to β ($\alpha \ll \beta$) means that if $\epsilon > 0$, then there is $\delta > 0$ such that if $|\beta|_s(E) < \delta$, then $|\alpha|_s(E) < \epsilon$.

The statement that α is singular with respect to $\beta(\alpha \perp \beta)$ means that if $\epsilon > 0$, then there exists $E \in \Sigma$ such that $|\beta|_s(E) < \epsilon$ and $|\alpha|_s(S-E) < \epsilon$.

Denote by $L(B, A)$ the space of bounded linear transformations from B to A.

The statement that T is a (Σ, M)-simple function means that T is a function on S to M and, moreover, there is a finite partition $\pi = \{E_i\}_{i \leq n}$ of S in Σ (i.e., $E_i \in \Sigma$, $i \leq n$) such that T is constant on the elements E_i of π.

Henceforth, suppose that A has a basis $\{a_i\}$ of unit vectors.

THEOREM 4. *Suppose that A is n-dimensional, that B is m-dimensional, and that $\alpha \ll \beta$. Then there is a sequence T_k of $(\Sigma, L(B, A))$-simple functions such that* $\lim_k |\alpha - \int T_k d\beta|_s(S) = 0$.

PROOF. From $\alpha \ll \beta$ and $|\alpha_i(E)| \leq 2M_A \|\alpha(E)\|$ it follows that $\alpha_i \ll \beta$. Moreover, letting $\alpha_{i1} = \alpha_i \wedge \beta_1$, where $\alpha_i \wedge \beta_1$ denotes that part of α_i which is absolutely continuous with respect to β_1 (see e.g. [2]), and $\alpha_{ij} = (\alpha_i - \sum_{k<j} \alpha_{ik}) \wedge \beta_j$ for $i < j \leq m$, it follows that $\alpha_{ij} \perp \alpha_{ik}$ if $j \neq k$ and also that $\alpha_i = \sum_{j \leq m} \alpha_{ij}$. For $i \leq n$ and $j \leq m$ there is a sequence $\{f_{ijk}\}_{k \geq 1}$ of (Σ, R)-simple functions such that

$$\lim_k v(\alpha_{ij} - \int f_{ijk} d\beta_j, S) = 0.$$

Let $T_k(x)$ be the transformation whose matrix with respect to the bases $\{a_i\}$ and $\{b_j\}$ is given by $(f_{ijk}(x))_{i,j}$. Then T_k is a sequence of $(\Sigma, L(B, A))$ simple functions and $\lim_k |\alpha - \int T_k d\beta|_s(S) = 0$.

In the next theorem it will be convenient to have the following notation. For each positive integer m, let β^m be defined by $\beta^m(E) = \sum_{i \leq m} b_i \beta_i(E)$, $E \in \Sigma$. Then $\beta^m \in H(B^m)$ where B^m denotes the closed linear span, in B, of $b_1, \cdots, b_m$.

THEOREM 5. *Suppose that* $\lim_m |\beta - \beta^m|_s(S) = 0$ *and, moreover, that* $\alpha \ll \beta$. *Then there is an increasing sequence* $\{m_k\}$ *of positive integers and a sequence* T_k *of* $(\Sigma, L(B^{m_k}, A^k))$*-simple functions such that* $|\alpha^k - \int T_k d\beta|_s(S) < 1/k$, *where* $\int T_k d\beta$ *is defined to be* $\int T_k d\beta^{m_k}$.

PROOF. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that if $|\beta|_s(E) < \delta$, then $|\alpha|_s(E) < \epsilon$. Let m satisfy $|\beta - \beta^m|_s(S) < \delta/2$. Then $|\beta^m|_s(E) < \delta/2$ implies that $|\alpha|_s(E) < \epsilon$. Notice that $|\beta|_s(E) < \delta/(2M_B)$ implies that $|\beta^m|_s(E) < \delta/2$, and that if $|\alpha|_s(E) < \epsilon$, then $|\alpha_i|_s(E) < M_A \epsilon$. For $k \geqq 1$, choose ϵ_k so that $M_A \epsilon_k < 1/(4k^2)$. Then choose $\delta_k (\delta_k < \delta_{k-1})$ and $m_k (m_k > m_{k-1})$ in accordance with the preceding statements. For $i \geqq 1$, let $\alpha_{i1} = \alpha_i \wedge \beta_1$ and, for $j > 1$, let $\alpha_{ij} = (\alpha_i - \sum_{p<j} \alpha_{ip}) \wedge \beta_j$. Let $\gamma_{ij} = \sum_{p \leq j} \alpha_{ip}$. Then $\alpha_{ij} \perp \alpha_{ip}$ if $j \neq p$, and $(\alpha_i - \gamma_{ij}) \perp \beta_p$ for $p \leqq j$. Hence if $\lambda > 0$ and $p \leqq j$, then there exists $E_p \in \Sigma$ such that $v(\beta_p, E_p) < \lambda$ and $v(\alpha_i - \gamma_{ij}, S - E_p) < \lambda$. If $F = \bigcap_{p \leq j} E_p$, then $v(\alpha - \gamma_{ij}, S - F) < j\lambda$ and $v(\beta_p, F) < \lambda$, $p \leqq j$, which implies that $|\beta^j|_s(F) \leqq j\lambda$. If $j = m_k$ and λ satisfies $m_k \lambda < \min(\delta_k/2, 1/(4k^2))$, then $v(\alpha_i - \gamma_{im_k}, S - F) < 1/(4k^2)$ and $|\beta^{m_k}|_s(F) < \delta_k/2$. This latter inequality, together with the heretofore assigned properties of δ_k, implies that $|\alpha_i|_s(F) < 1/(4k^2)$ and, in turn, that $v(\alpha_i, F) < 1/(2k^2)$. Then, since $v(\alpha_i, E) = v(\alpha_i - \gamma_{ij}, E) + v(\gamma_{ij}, E)$, $E \in \Sigma$, $i, j \geqq 1$, it follows that $v(\alpha_i - \gamma_{im_k}, F) < 1/(2k^2)$ and, hence, $v(\alpha_i - \gamma_{im_k}, S) < 3/(4k^2)$ for $i \geqq 1$. Because $\alpha_{ij} \ll \beta_j$ there are (Σ, R)-simple functions f_{ijk} such that $v(\alpha_{ij} - \int f_{ijk} d\beta_j, S) < 1/(4k^2 m_k)$. Thus $v(\gamma_{im_k} - \int \sum_{j \leq m_k} f_{ijk} d\beta_j, S) < 1/(4k^2)$ and, hence, $v(\alpha_i - \int \sum_{j \leq m_k} f_{ijk} d\beta_j, S) < 1/k^2$. From this latter inequality it follows that if $T_k(x)$ is the matrix whose (i, j)-th entry is $f_{ijk}(x)$ for $i \leqq k$ and $j \leqq m_k$ (and zero otherwise, when necessary, in order that T_k have the proper domain and range), then $|\alpha^k - \int T_k d\beta|_s(S) \leqq 1/k$.

COROLLARY 5.1. *For each positive integer* n, $\lim_k |\alpha^n - \int T_k d\beta|_s(S) = 0$.

COROLLARY 5.2. *For each* $E \in \sum$, $\alpha(E) = \lim_k \int_E T_k d\beta$.

COROLLARY 5.3. *If* $\lim_n |\alpha - \alpha^n|_s(S) = 0$, *then* $\lim_k |\alpha - \int T_k d\beta|_s(S) = 0$.

Example 2 shows that norm convergence of the sequence $\{\beta_i\}$ to zero is not a sufficient condition in order that $\lim|\beta-\beta^n|_s(S)=0$. The following example shows that $|||\beta|||<\infty$ is not a necessary condition in order that $\lim|\beta-\beta^n|_s(S)=0$, where $|||\beta|||=\sup\{\Sigma\|\beta(E_i)\|;\ \pi=\{E_i\}$ is a finite partition of S comprised of elements of $\Sigma\}$.

EXAMPLE 3. Suppose that S is the set of positive integers, Σ is the algebra of all subsets of S, $B=l_2$, $b_i=\{\delta_{ij}\}_{j\geqq 1}$, $\beta_i(E)=(1/i)\sum_{j\in E}\delta_{ij}$, and $\beta(E)=\sum b_i\beta_i(E)$. Then $\beta\in H(B)$, $\{b_i\}$ satisfies condition G_2, $|||\beta|||=\infty$ and $\lim|\beta-\beta^n|_s(S)=0$.

We conclude with conditions which are sufficient to insure that $\lim|\beta-\beta^n|_s(S)=0$.

THEOREM 6. *Suppose that $|||\beta|||=M<\infty$ and that $\{b_i\}$ satisfies condition G_2. Moreover suppose that if*

$$x_k=\sum_{i\leqq n}b_i\lambda_{ki}\quad\text{and}\quad y_k=\sum_{n<i\leqq m_k}b_i\lambda_{ki},\qquad k\leqq m,$$

then

$$\left\|\sum_{k\leqq m}(x_k+y_k)\right\|-\left\|\sum_{k\leqq m}x_k\right\|\leqq\sum_{k\leqq m}(\|x_k+y_k\|-\|x_k\|).$$

Then $\lim|\beta-\beta^n|_s(S)=0$.

PROOF. Let $\epsilon>0$. Then let δ be a positive number such that if $\|\sum_{i\leqq n}b_i\lambda_i\|\leqq M$ and $\|\sum_{n<i\leqq m}b_i\lambda_i\|\geqq\epsilon$, then $\|\sum_{i\leqq m}b_i\lambda_i\|\geqq\|\sum_{i\leqq n}b_i\lambda_i\|+\delta$. Let $\pi=\{E_i\}$ be a finite partition of S in Σ satisfying $\sum_i\|\beta(E_i)\|>M-\delta/2$. Let n be a positive integer satisfying $\sum_i\|(\beta-\beta^k)(E_i)\|<\delta/2$ if $k\geqq n$, and let $k\geqq n$. Then $\sum_i\|\beta^k(E_i)\|>M-\delta$. Suppose that $|\beta-\beta^k|_s(S)>\epsilon$. Then there is an element E of Σ satisfying $\|(\beta-\beta^k)(E)\|>\epsilon$ and, hence, $(\|\beta(E)\|-\|\beta^k(E)\|)\geqq\delta$. Let π_1 be the partition of S generated by π and E, let $F_i=E_i\cap E$ and let $G_i=E_i-E$. Then the contradiction

$$\begin{aligned}\sum_{T\in\pi_1}\|\beta(T)\|&=\sum_{T\in\pi_1}\|\beta^k(T)\|+\sum_i(\|\beta(F_i)\|-\|\beta^k(F_i)\|)\\&\quad+\sum_i(\|\beta(G_i)\|-\|\beta^k(G_i)\|)\geqq\sum_{E\in\pi}\|\beta^k(E)\|\\&\quad+(\|\beta(E)\|-\|\beta^k(E)\|)>(M-\delta)+\delta=M\end{aligned}$$

obtains.

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