

ON CONTINUITY IN TWO VARIABLES

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1. Introduction. It is well known that a real-valued function can be continuous in each fixed variable on the product of two intervals and yet fail to be continuous while, if a function is complex analytic in each fixed variable on the product of two open disks, then the function is complex analytic and hence continuous. The purpose of this note is to consider an intermediate case.

Let f be a complex-valued function defined on a bicylinder $B = B_1 \times B_2$ with the property that $f(z, \cdot)$ is continuous for fixed $z \in B_1$ and $f(\cdot, w)$ is holomorphic for fixed $w \in B_2$. We shall say that $f \in H$ on B if these conditions hold. In [3], we asked whether or not such a function f must be continuous, thus, whether or not Hartogs' classical theorem could be generalized to this extent. In this note we shall give an example to show that f need not be continuous and we also give some conditions which are sufficient to insure the continuity of f .

2. Counterexample. Before exhibiting the example we need the following result.

LEMMA. *Let $f \in H$ on $B = B_1 \times B_2$. Then f is continuous on B if and only if each sequence of functions $\{f(\cdot, w_n)\}$, where $\lim w_n = w_0 \in B_2$, converges almost uniformly on B_1 .*

PROOF. "if." This follows from the equivalence of continuous convergence and uniform convergence. See [1].

"only if." This follows from uniform continuity.

Now let B be the unit bicylinder, $B = \{(z, w) : |z| < 1, |w| < 1\}$. Using Runge's theorem there exists a sequence of polynomials $\{P_n(z)\}$ which converge to zero on $|z| < 1$ but the convergence is not uniform on any neighborhood of any point of the real axis in $|z| < 1$. Using this sequence we define a function F on B as follows:

(i) on the circles in $|w| < 1$ of radius $1/n$, $n = 2, 3, \dots$, let $F(z, w) = P_n(z)$. Let $F(z, 0) \equiv 0$.

(ii) For w , with $1/(n+1) < |w| < 1/n$, $n = 1, 2, \dots$, consider the circles C_t with radius $r = t/n + (1-t)/(n+1)$, $0 < t < 1$. For $w \in C_t$ define

$$F(z, w) = tP_n(z) + (1 - t)P_{n+1}(z).$$

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Then $F \in H$ on B but is not continuous by virtue of the lemma.

3. Another characterization. Let $F \in H$ on the unit bicylinder. Define the "radius of continuity" as follows. For z_0 , with $|z_0| < 1$, $R(z_0)$ is the distance from $w=0$ to the nearest point w_0 such that f is discontinuous at (z_0, w_0) . Using a method due to Rothstein [2] we shall prove the following theorem.

THEOREM. *Let $f \in H$ on B , where B is the unit bicylinder. Then f is continuous if and only if R is superharmonic.*

For the proof we shall need the following results.

PROPOSITION 1. *Let S be a complete metric space and F a family of continuous real-valued functions on S . Then, if for each $\chi \in S$, the set $\{f(\chi) : f \in F\}$ is bounded, there exists a closed sphere U of positive radius in S and a number M such that $|f(\chi)| \leq M$ for all $f \in F$ and all $\chi \in U$.*

PROOF. This is well known.

PROPOSITION 2. *Let h be superharmonic in $|z| < 1$ with $h \geq 0$. Let S_n, S, T_n be points in $|z| < \alpha < 1$ with $S_n \rightarrow S$ and $|T_n - S| > d > 0$. Suppose J_n is a Jordan arc joining S_n to T_n in $|z| < \alpha$. Then if $h(z) \geq r$ on all J_n , we have $h(S) \geq r$.*

PROOF. The proof is in [2].

Now let $D(k)$ denote a domain in $|z| < 1$ for which $R(z) \geq k$ and which is maximal with respect to this property.

LEMMA 1. *Given k such that $0 < k < 1$, there exists a $D(k)$.*

PROOF. We need only show that there exists an open disk D contained in $|z| < 1$ such that f is continuous on $D \times \{|w| \leq k\}$.

To see this we apply Proposition 1 to the family $\{g_w(z)\}$, where

$$g_w(z) = |f(z, w)|, \quad |z| \leq 1/2, \quad |w| < k', \quad k < k' < 1.$$

Hence there exists an open disk D in $|z| < 1/2$ and a constant K such that $|f| \leq K$ on $D \times \{|w| < k'\}$. Then using Vitali's theorem and the lemma of §2 we get our result.

Note that if $f \in H$ on B and is locally bounded, then f is continuous.

LEMMA 2. *If q is a boundary point of $D(k)$ and $|q| < 1$, then there exists a sequence $q_n \rightarrow q$ with $R(q_n) < k$.*

PROOF. Clear.

LEMMA 3. *Let $\{D_n(k)\}$ be a sequence of domains of type $D(k)$. Let $z_0, |z_0| < 1$, be a limit point of the $D_n(k)$ and let d_n be the diameter of $D_n(k)$. If $R(z_0) < k$, then $\lim d_n = 0$.*

PROOF. Apply Proposition 2.

PROOF OF THE THEOREM. The “only if” part of the theorem is trivial. For the “if” part, let k be such that $0 < k < 1$. We know by Lemma 1 that there exists a $D(k)$ in $|z| < 1$. Suppose this $D(k)$ has a boundary point z_1 , with $|z_1| < 1$. Let K_1 be a closed disk in $|z| < 1$ with center z_1 and let \bar{M} be the set of all boundary points q of all domains of type $D(k)$, where $q \in \text{int } K_1$, and let \bar{M} be the closure of this set. Using Proposition 1 on $\bar{M} \cap K_1$ there exists a closed disk $K_2 \subset K_1$ with center $z_2 \in \bar{M}$ and a number $m > 0$ such that $|f(z, w)| \leq m$ for $z \in \bar{M} \cap K_2$ and $|w| < k$.

By Lemma 2, there exists in K_2 a point z_3 with $z_3 \in \text{int } K_2$ and $R(z_3) = k_1 < k$. Now let $d > 0$ be chosen small enough that the boundaries of all $D(k)$ which have points in common with $K_3 = \{ |z - z_3| \leq d \}$ are completely contained in $\bar{M} \cap K_2$. This is possible by Lemma 3. Then by the maximum principle we have $|f(c, w)| \leq m$ for $|w| < k$ and all $c \in D'(k)$ where $D'(k)$ is any domain of type $D(k)$ which intersects K_3 . But using Proposition 1 we see that these intersections must be dense in K_3 and hence f is bounded and thus is continuous on $\text{int } K_3 \times \{ |w| < k \}$. This contradicts the fact that $R(z_3) = k_1 < k$ and proves the result.

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