ON THE SHEFFER A-TYPE OF CERTAIN MODIFIED POLYNOMIAL SETS

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1. **Introduction.** Let $\{p_n^{(\alpha)}(x)\}$ be a simple polynomial set defined by a generating function of the form

$$(1)\qquad (1-t)^{-\alpha}F(x,t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)t^n$$

where $F(x, t)$ is independent of the parameter α. The object of this paper is to examine certain fundamental properties of the set $\{p_n(x)\}$, the case when $\alpha=0$, that are inherited by $\{p_n^{(\alpha)}(x)\}$ and, more generally, by the *modified* set $\{p_n^{(\alpha+\beta n)}(x)\}$.

A variety of familiar polynomial sets possess generating functions of the form (1). For example, if $F(x,t)=(1-t)^{-1}\exp[-xt/(1-t)]$, we have the generalized Laguerre polynomials $p_n^{(\alpha)}(x)=L_n^{(\alpha)}(x)$ [7, p. 202]. Indeed, this paper was initially motivated by a desire to study the basic structure of a certain property of the simple Laguerre set $\{L_n(x)\}$ $(\alpha=0)$ which is preserved by $\{L_n^{(\alpha+\beta n)}(x)\}$ (see remark following Theorem 2). This modified Laguerre set has appeared scattered throughout the literature for special cases of β. For example, Toscano [9] initiated serious study of the case $\beta=1$; the cases $\beta=0$, -1 are perhaps the most familiar; and Al-Salam [1] has shown the case $\beta=-2$ to be essentially the set of reversed Bessel polynomials.

All of our results are in the language of formal power series and are consequences of the following

LEMMA. *The set $\{p_n^{(\alpha+\beta n)}(x)\}$ is generated by*

$$(2)\qquad \frac{[1-u(t)]^{1-\alpha}}{1-(1+\beta)u(t)}F(x,u(t)) = \sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x)t^n$$

where $u(t)$ is the inverse of $v(t)=t(1-t)^{\beta}$. That is, $v(u(t))=u(v(t))=t$.

PROOF. In view of

$$(1-t)^{-\alpha}F(x,t) = \left\{\sum_{n=0}^{\infty}\binom{-\alpha}{n}(-1)^n t^n\right\}\left\{\sum_{n=0}^{\infty} p_n(x)t^n\right\}$$

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it is immediate that

$$p_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{-\alpha}{n-k} (-1)^{n-k} p_k(x)$$

and so

$$\sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x) t^n = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{-\alpha-\beta n}{n-k} (-1)^{n-k} p_k(x) \right\} t^n$$
$$= \sum_{k=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \binom{-\alpha-\beta k-\beta n}{n} (-t)^n \right\} p_k(x) t^k.$$

Next recall the identity [6, p. 302]

$$\sum_{n=0}^{\infty} \binom{a+bn}{n} \left[\frac{z}{(1+z)^b} \right]^n = \frac{(1+z)^{1+a}}{1+(1-b)z}$$

and set $a = -\alpha-\beta k$, $b = -\beta$, and $z = -u(t)$ where $u(t)$ is as defined in the statement of the lemma. This yields

$$\sum_{n=0}^{\infty} \binom{-\alpha-\beta k-\beta n}{n} (-t)^n = \frac{[1-u(t)]^{1-\alpha-\beta k}}{1-(1+\beta)u(t)}$$

which means that

$$\sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x) t^n = \frac{[1-u(t)]^{1-\alpha}}{1-(1+\beta)u(t)} \sum_{k=0}^{\infty} p_k(x) (t/[1-u(t)]^{\beta})^k.$$

But $\sum_{k=0}^{\infty} p_k(x)(t/[1-u(t)]^{\beta})^k = F(x, \ t/[1-u(t)]^{\beta}) = F(x,u(t))$, and we arrive at (2).

2. **Generalized Appell sets of finite A-type.** Suppose $\{p_n(x)\}$ is what Boas and Buck [2, p. 18] have defined to be a generalized Appell set. That is, it is generated by

$$G(t)\Phi[xH(t)] = \sum_{n=0}^{\infty} p_n(x) t^n \tag{3}$$

where $G(t)$ and $H(t)/t$ possess power series expansions with nonzero initial coefficients and the expansion of $\Phi(t)$ has all nonzero coefficients. With $F(x, t) = G(t)\, \Phi[xH(t)]$ in our lemma then, we have

$$\frac{[1-u(t)]^{1-\alpha}}{1-(1+\beta)u(t)} G(u(t))\Phi[xH(u(t))] = \sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x) t^n \tag{4}$$

and the form of this generating function yields

THEOREM 1. *If* $\{p_n(x)\}$ *is a generalized Appell set, then so is* $\{p_n^{(\alpha+\beta n)}(x)\}$.

Now the class of generalized Appell sets contains the subclass of A-type zero sets introduced by Sheffer [8]. Specifically, to say that $\{p_n(x)\}$ is of A-type zero is to say that there exists a differential operator $J=\sum_{k=0}^{\infty} j_k D^{k+1}$, where the j_k are constants, such that $J[p_n(x)]=p_{n-1}(x)$ for all $n\geqq 1$. According to Sheffer's alternative characterization in terms of generating functions, $\{p_n(x)\}$ is of A-type zero iff in (3) $\Phi(t)=\exp(t)$. That is,

$$G(t)\exp[xH(t)] = \sum_{n=0}^{\infty} p_n(x)t^n \tag{5}$$

in which case (4) reduces to

$$\frac{[1-u(t)]^{1-\alpha}}{1-(1+\beta)u(t)} G(u(t))\exp[xH(u(t))] = \sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x)t^n, \tag{6}$$

and we have

THEOREM 2. *If* $\{p_n(x)\}$ *is a generalized Appell set of A-type zero, then so is* $\{p_n^{(\alpha+\beta n)}(x)\}$.

REMARK. The operator $J=\sum_{k=0}^{\infty} j_k D^{k+1}$ associated with $\{p_n(x)\}$ is said to be generated by the function $J(t)=\sum_{k=0}^{\infty} j_k t^{k+1}$, and Sheffer pointed out that $J(t)$ is the inverse of the function $H(t)$ in (5). Analogously from (6), we find that $\{p_n^{(\alpha+\beta n)}(x)\}$ corresponds to the operator generated by the inverse of $H(u(t))$. This inverse is evidently $v(J(t)) =J(t)[1-J(t)]^{\beta}$. Now Sheffer observed the simple Laguerre set $\{L_n(x)\}$ to be of A-type zero corresponding to the operator generated by $-t/(1-t)$. Inasmuch as $\{L_n(x)\}$ is a special case of $\{p_n(x)\}$, the above discussion reveals that $\{L_n^{(\alpha+\beta n)}(x)\}$ is of A-type zero corresponding to the operator generated by $-t/(1-t)^{1+\beta}$. The modified Laguerre set is already known to be of A-type zero [3], [4]; but the function generating the associated operator has been stated explicitly for nonzero β only when $\beta=1$ [7, p. 299].

While Sheffer concentrated on A-type zero sets, he did extend the concept to include sets of positive A-type. The set $\{p_n(x)\}$ is of A-type $m\geqq 0$ iff there exists a differential operator $J=\sum_{k=0}^{\infty} j_k(x)D^{k+1}$ such that the $j_k(x)$ are polynomials with maximum degree m and $J[p_n(x)]=p_{n-1}(x)$ for all $n\geqq 1$. Suppose now that our $\{p_n(x)\}$ is a generalized Appell set of some positive A-type m. Goldberg [5] has shown that this is equivalent to saying the generalized Appell generating function in (3) takes the generalized hypergeometric form

$$(7) \qquad G(t)\,{}_0F_q\left(\begin{array}{c|c} \overline{} & \\ \overline{b_q} & cxH(t)\end{array}\right) = \sum_{n=0}^{\infty} p_n(x)t^n$$

where q is a positive integer dividing m, c is some nonzero constant, and the inverse $J^*(t)$ of $H(t)$ is a polynomial of degree m/q. It is, of course, understood that none of the denominator parameters of the ${}_0F_q$ is zero or a negative integer. Equation (4) takes the corresponding form

$$(8) \qquad \frac{[1-u(t)]^{1-\alpha}}{1-(1+\beta)u(t)}\, G(u(t))\,{}_0F_q\left(\begin{array}{c|c} \overline{} & \\ \overline{b_q} & cxH(u(t))\end{array}\right) = \sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x)t^n,$$

and we ask if $\{p_n^{(\alpha+\beta n)}(x)\}$ is of some finite A-type. It clearly cannot be of A-type zero because $q\neq 0$ and the ${}_0F_q$ cannot therefore be an exponential. If $\{p_n^{(\alpha+\beta n)}(x)\}$ is of some positive A-type, the function $v(J^*(t))=J^*(t)[1-J^*(t)]^\beta$ which is the inverse of $H(u(t))$ must be a polynomial of positive degree. This degree would be $m/q+(m/q)\beta=(1+\beta)m/q$ and β would have to be a nonnegative integer. Now if β is in fact a nonnegative integer, we see that (8) takes the proper form for $\{p_n^{(\alpha+\beta n)}(x)\}$ to be of positive A-type, specifically $(1+\beta)m$. In summary, we have

THEOREM 3. *If $\{p_n(x)\}$ is a generalized Appell set of positive A-type m, then a necessary and sufficient condition for $\{p_n^{(\alpha+\beta n)}(x)\}$ to be of some finite A-type is that β be a nonnegative integer, in which case the latter A-type is $(1+\beta)m$.*

REMARK. Goldberg pointed out that the polynomial $J^*(t)=\sum_{k=0}^{(m/q)-1} j_k^* t^{k+1}$ generates the positive A-type operator J in the sense that $J=\sum_{k=0}^{(m/q)-1} j_k^*\sigma^{k+1}$ where $\sigma=D\prod_{i=1}^{q}(xD+b_i-1)$. The b_i are the parameters in the denominator of the ${}_0F_q$ in (7). So when $\{p_n^{(\alpha+\beta n)}(x)\}$ is of positive A-type, the associated operator is generated by $J^*(t)[1-J^*(t)]^\beta$ with the same σ.

3. **Remarks on sets of finite B- and C-type.** Sheffer went on to discuss sets of B- and C-type m where m is a nonnegative integer. We do not restate his original definitions but simply recall that he showed a polynomial set to be of B-type m iff it is of C-type m. In fact, to say that $\{p_n(x)\}$ is of B- or C-type m is to say that it possesses a generating function of the form

$$(9) \qquad G(t)\exp[xH_0(t)+x^2H_1(t)+\cdots+x^{m+1}H_m(t)] = \sum_{n=0}^{\infty} p_n(x)t^n.$$

Here the functions $G(t)$ and $H_k(t)/t^{k+1}$ $(k=0, 1, \cdots, m)$ have power series representations where the initial coefficients of at least $G(t)$ and $H_0(t)/t$ are nonzero. As Sheffer pointed out, comparison of (9) when $m=0$ with (5) shows that the A-, B-, and C-type zero classes actually coincide. It follows from our lemma that

$$\begin{aligned}\frac{[1-u(t)]^{1-\alpha}}{1-(1+\beta)u(t)} G(u(t)) \exp[xH_0(u(t)) + x^2H_1(u(t)) + \cdots \\ + x^{m+1}H_m(u(t))] = \sum_{n=0}^{\infty} p_n^{(\alpha+\beta n)}(x)t^n,\end{aligned} \tag{10}$$

and with this we find

THEOREM 4. *If* $\{p_n(x)\}$ *is a set of finite B- or C-type m, then so is* $\{p_n^{(\alpha+\beta n)}(x)\}$.

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