INFINITE SYSTEMS OF NONLINEAR OSCILLATION EQUATIONS RELATED TO THE STRING[1]

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1. Introduction. The purpose of this paper is to discuss the existence and uniqueness of solutions to an infinite system of nonlinear oscillation equations of the form

$$(1.1)\qquad T_j'' + j^2\left(a_0 + a_1 \sum_{l=1}^{\infty} l^2 T_l^2\right) T_j = 0, \qquad j = 1, 2, \cdots, \infty,$$

where the constants a_0 and a_1 satisfy the conditions $a_0 \geqq 0$ and $a_1 > 0$ (the prime in (1.1) indicates differentiation with respect to t). The initial conditions on (1.1) will be taken as

$$(1.2\text{a})\qquad T_j(0) = \alpha_j,$$

$$(1.2\text{b})\qquad T_j'(0) = \beta_j.$$

Equations of the type (1.1) are related to the Duffing equation (cf. [1]), and arise in attempting to find Fourier series solutions

$$(1.3)\qquad W(x, t) = \sum_{j=1}^{\infty} T_j(t) \sin j\pi x/L,$$

to the nonlinear partial integro-differential equation

$$(1.4)\qquad W_{tt} - \left(H_0 + H_1 \int_0^L W_\xi(\xi, t)^2 d\xi\right) W_{xx} = 0,$$

($H_0 \geqq 0$, $H_1 > 0$). Equation (1.4) describes the small amplitude vibrations of a string in which the dependence of the tension on the deformation cannot be neglected (cf. [2], [3]).

The equations (1.1) form an infinite Hamiltonian system, and in fact there is no difficulty in showing that any solution of (1.1) satisfies the condition

$$(1.5)\qquad \frac{d}{dt}\left(\sum_{j=1}^{\infty} (T_j')^2 + a_0 \sum_{j=1}^{\infty} j^2 T_j^2 + \frac{a_1}{2}\left(\sum_{j=1}^{\infty} j^2 T_j^2\right)^2\right) = 0.$$

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At first glance it would appear that if the initial conditions (1.2) satisfy a finite energy condition, i.e.,

$$h = \sum_{j=1}^{\infty} \beta_j^2 + a_0 \sum_{j=1}^{\infty} j^2 \alpha_j^2 + \frac{a_1}{2}\left(\sum_{j=1}^{\infty} j^2 \alpha_j^2\right)^2 < \infty, \tag{1.6}$$

then (1.1) should have a solution for all $t>0$. Indeed this is the case for finite systems of the form (1.1) since the finite system

$$T_j'' + j^2\left(a_0 + a_1 \sum_{l=1}^{N} l^2 T_l^2\right) T_j = 0, \qquad j = 1, 2, \cdots, N, \tag{1.7}$$

has associated with it a Lipschitz constant (depending on N). Thus the method of successive approximation (cf. [4]) may be used to show the existence of a solution to (1.7) locally, and the continuation of this solution is guaranteed by the fact that the energy—and hence the solution and its derivative—remains bounded. However, the infinite system of equations (1.1) is not Lipschitz continuous because of the unbounded nature of the coefficient of T_j as $j\to\infty$. Thus the method of successive approximation fails and an alternative procedure is necessary.

In §2 of this paper, it will be shown that under certain conditions on the initial data (1.2), solutions of the finite system (1.7) converge to a solution of (1.1) as $N\to\infty$. In order to guarantee this it will be necessary to require that the initial data (1.2) satisfies a condition stronger than the simple finite energy condition (1.6). In §3 it will be shown that the solution of (1.1) satisfying the initial conditions (1.2) is unique among a certain class of functions.

2. **Existence.** In proving the existence of solutions to (1.1), it is convenient to define a set of functions $T_{j,N}$ as follows: for $j \leqq N$, $T_{j,N}$ is to be a solution of the finite system of equations (1.7) and satisfy the initial conditions (1.2) for $j=1, 2, \cdots, N$, and for $j>N$ set $T_{j,N}\equiv 0$. The functions $T_{j,N}$ are of course also solutions of the infinite system (1.1), i.e.

$$T_{j,N}'' + j^2 A_N T_{j,N} = 0, \qquad j = 1, 2, \cdots, \infty, \tag{2.1}$$

where

$$A_N = a_0 + a_1 \sum_{l=1}^{\infty} l^2 T_{l,N}^2. \tag{2.2}$$

If in addition the initial data (1.2) satisfies the finite energy condition (1.6) it follows that (cf. (1.5))

$$\sum_{j=1}^{\infty} (T'_{j,N})^2 + a_0 \sum_{j=1}^{\infty} j^2 T_{j,N}^2 + \frac{a_1}{2}\left(\sum_{j=1}^{\infty} j^2 T_{j,N}^2\right)^2 \leqq h. \tag{2.3}$$

Thus there exist constants M_1 and M_2 independent of N such that

$$\sum_{j=1}^{\infty} (T'_{j,N})^2 \leqq M_1, \tag{2.4a}$$

$$\sum_{j=1}^{\infty} j^2 T_{j,N}^2 \leqq M_2. \tag{2.4b}$$

It is a consequence of (2.4b) that the functions A_N are uniformly bounded independent of N. If it could also be shown that $|A'_N|$ was uniformly bounded independent of N—so that the sequence $\{A_N\}$ is not only bounded but equicontinuous—the existence of a uniformly convergent subsequence would follow from the Arzela lemma (cf. [4]). Indeed the demonstration of the uniform boundedness of $|A'_N|$ is the key step in proving the existence of solutions to (1.1). In what follows it will be necessary to assume that the initial data satisfies the conditions

$$\sum_{j=1}^{\infty} j^2 \beta_j^2 < \infty, \tag{2.5a}$$

$$\sum_{j=1}^{\infty} j^4 \alpha_j^2 < \infty. \tag{2.5b}$$

This requirement on the initial data is, of course, stronger than the energy condition (1.6).

LEMMA 2.1. *If the initial data* (1.2) *satisfies the condition* (2.5), *and*

$$a_0 + a_1 \sum_{j=1}^{\infty} j^2 \alpha_j^2 \neq 0, \tag{2.6}$$

there exists an interval $0 \leqq t < t_c$, *such that* $|A'_N|$ *is uniformly bounded independent of* N *on any closed subinterval* $0 \leqq t \leqq t^* < t_c$.

PROOF. After differentiating the function A_N, the Schwarz inequality yields

$$\begin{aligned} |A'_N| &\leqq 2a_1 \sum_{l=1}^{\infty} l^2 |T_{l,N}| |T'_{l,N}| \\ &\leqq 2a_1 \left\{ \sum_{l=1}^{\infty} l^2 T_{l,N}^2 \sum_{l=1}^{\infty} l^2 (T'_{l,N})^2 \right\}^{1/2} \\ &\leqq 2 \left\{ a_1 A_N \sum_{l=1}^{\infty} l^2 (T'_{l,N})^2 \right\}^{1/2}. \end{aligned} \tag{2.7}$$

Thus the object is to estimate the functions $|T'_{j,N}|$. For this purpose define a function $E_{j,N}$ as (cf. [5])

$$E_{j,N} = \frac{(T'_{j,N})^2}{j^2 A_N} + T_{j,N}^2 \geqq 0. \tag{2.8}$$

The condition (2.6) guarantees that $E_{j,N}$ is defined in some neighborhood of $t=0$ when N is sufficiently large (the condition (2.6) is equivalent to the requirement that the partial differential equation (1.4) be hyperbolic at $t=0$). The functions $T_{j,N}$ are solutions of (2.1); therefore differentiation of (2.8) yields

$$E'_{j,N} = -\frac{A'_N}{A_N}\left(\frac{(T'_{j,N})^2}{j^2 A_N}\right) \leqq \frac{|A'_N|}{A_N} E_{j,N}, \tag{2.9}$$

or equivalently

$$E_{j,N} \leqq E_{j,N}(0) \exp\left(\int_0^t \frac{|A'_N|}{A_N}\, d\tau\right). \tag{2.10}$$

Estimates on both $T_{j,N}$ and $T'_{j,N}$ follow from (2.10). Thus it is found that

$$T_{j,N}^2 \leqq (\beta_j^2/j^2 A_N(0) + \alpha_j^2) \exp\left(\int_0^t \frac{|A'_N|}{A_N}\, d\tau\right), \tag{2.11a}$$

$$(T'_{j,N})^2/j^2 A_N \leqq (\beta_j^2/j^2 A_N(0) + \alpha_j^2) \exp\left(\int_0^t \frac{|A'_N|}{A_N}\, d\tau\right). \tag{2.11b}$$

Define K_N as

$$K_N = \sum_{j=1}^{\infty} (j^2\beta_j^2/A_N(0) + j^4\alpha_j^2), \tag{2.12}$$

and note that finiteness of (2.12) follows from (2.5). In addition, the fact that $A_N(0) \leqq A_{N+1}(0)$ shows that

$$K_{N+1} \leqq K_N. \tag{2.13}$$

Combining (2.11b) and (2.7) it follows that

$$|A'_N| \leqq 2\left\{a_1 A_N^2 K_N \exp\left(\int_0^t \frac{|A'_N|}{A_N}\, d\tau\right)\right\}^{1/2}, \tag{2.14}$$

or

$$(2.15)\qquad -\frac{d}{dt}\exp\left(-\frac{1}{2}\int_0^t \frac{|A_N'|}{A_N}\,d\tau\right) \leqq (a_1K_N)^{1/2} \leqq (a_1K_M)^{1/2},$$

if $N \geqq M$. It is a consequence of (2.15) that

$$(2.16)\qquad \exp\left(\frac{1}{2}\int_0^t \frac{|A_N'|}{A_N}\,d\tau\right) \leqq 1/(1-(a_1K_M)^{1/2}t)$$

for all $N \geqq M$ and all t in the interval

$$(2.17)\qquad 0 \leqq t < t_M = 1/(a_1K_M)^{1/2}.$$

Combining (2.14) and (2.16), it is clear that when $N \geqq M$ and t is in the interval (2.17), $|A_N'|$ satisfies the bound

$$(2.18)\qquad |A_N'| \leqq 2(a_1K_M)^{1/2}A_N/(1-(a_1K_M)^{1/2}t).$$

Since A_N is uniformly bounded independent of N, (2.18) shows that $|A_N'|$ is also uniformly bounded independent of N in the interval (2.17). In fact, if K is defined as

$$(2.19)\qquad K = \lim_{N\to\infty} K_N$$

and t_c is defined as

$$(2.20)\qquad t_c = 1/(a_1K)^{1/2}$$

then $|A_N'|$ will be uniformly bounded in any closed interval $0 \leqq t \leqq t^* < t_c$. Q.E.D.

It is of interest to note that, at least in the case where $a_0 > 0$, the interval $0 \leqq t < t_c$ grows arbitrarily large as the initial data (1.2) approaches zero.

In view of the preceding remarks, Lemma 2.1 guarantees the existence of a subsequence $\{A_{N_i}\}$ which converges uniformly to a (continuous) function $A(t)$ on any closed subinterval $0 \leqq t \leqq t^* < t_c$. Let T_j be the solution of the (linear) equation

$$(2.21)\qquad T_j'' + j^2A(t)T_j = 0,$$

satisfying the initial conditions (1.2). There is no difficulty in showing that $T_{j,N_i} \to T_j$ and $T_{j,N_i}' \to T_j'$ on the interval $0 \leqq t \leqq t^* < t_c$. The existence of solutions to (1.1) is settled by the following

THEOREM 2.1. *The infinite system of equations* (1.1) *have a solution satisfying the initial data* (1.2) *on any closed interval* $0 \leqq t \leqq t^* < t_c$ *if the initial data satisfies the conditions* (2.5) *and* (2.6).

PROOF. It is only necessary to show that the solutions of the linear system (2.21) furnish a solution of the system (1.1). For this purpose it suffices to show that

$$A(t) = a_0 + a_1 \sum_{l=1}^{\infty} l^2 T_l^2. \tag{2.22}$$

The series which occurs in (2.22) converges since (cf. (2.11) and (2.16))

$$T_l^2 = \lim_{N_i \to \infty} T_{l,N_i}^2 \leqq (\beta_l^2/l^2 A(0) + \alpha_l^2)/(1 - a_1 K_M)^{1/2} t), \tag{2.23a}$$

$$(T_l')^2 = \lim_{N_i \to \infty} (T_{l,N_i}')^2 \leqq A(\beta_l^2/A(0) + l^2\alpha_l^2)/(1 - (a_1 K_M)^{1/2} t), \tag{2.23b}$$

for arbitrary M, and thus the series in (2.22) is majorized by a convergent series. The equality (2.22) follows from the estimate

$$\begin{aligned} \left| A - a_0 - a_1 \sum_{l=1}^{\infty} l^2 T_l^2 \right| &\leqq \left| A - A_{N_i} \right| \\ &+ a_1 \sum_{l=1}^{n} l^2 \left| T_l^2 - T_{l,N_i}^2 \right| + a_1 \sum_{l=n+1}^{\infty} l^2 (T_l^2 + T_{l,N_i}^2). \end{aligned} \tag{2.24}$$

The right side of (2.24) can be made arbitrarily small by first choosing n, then choosing N_i. Q.E.D.

The solution of (1.1) which has been constructed above has the properties

$$\sum_{j=1}^{\infty} j^4 T_j^2 < \infty, \tag{2.25a}$$

$$\sum_{j=1}^{\infty} j^2 (T_j')^2 < \infty, \tag{2.25b}$$

and

$$A(t) > 0 \tag{2.26}$$

in the interval $0 \leqq t < t_c$. The convergence conditions (2.25) are an immediate consequence of (2.23) and (2.5). The condition (2.26) follows from the fact that if $A(\eta) = 0$ for some value $t = \eta$ in the interval $0 \leqq t < t_c$, both $a_0 = 0$ and $T_j(\eta) = 0$ for all j (cf. (2.22)). If $T_j(\eta) = 0$ for all j the energy identity (1.5) shows that there exists at least one value of j such that $T_j'(\eta) \neq 0$. For this value of j, the inequality (2.23b) is violated at $t = \eta$. This contradiction proves (2.6).

3. **Uniqueness.** In this section it will be shown that the infinite system (1.1) has at most one solution satisfying the initial conditions (1.2) and the conditions (2.25a) and (2.26).

Assume T_j is a solution of (1.1) satisfying (2.25a) and (2.26) in some interval $0 \leqq t < \rho$. The function A (cf. (2.22)) is differentiable in the interval $0 \leqq t < \rho$ since the Schwarz inequality implies

$$|A'| \leqq 2a_1 \left\{ \sum_{l=1}^{\infty} l^4 T_l^2 \sum_{l=1}^{\infty} (T_l')^2 \right\}^{1/2}. \tag{3.1}$$

In view of (1.5) and (2.25a), both of the sums in (3.1) converge and are, in fact, uniformly bounded on any closed subinterval $0 \leqq t \leqq \rho^* < \rho$. In addition, the assumption that $A(t) > 0$ for $0 \leqq t < \rho$ implies that there exists a constant M such that

$$\exp\left(\frac{1}{2} \int_0^t \frac{|A'|}{A}\, d\tau\right) \leqq M \tag{3.2}$$

for $0 \leqq t \leqq \rho^* < \rho$.

Let T_j and S_j be solutions of (1.1) satisfying the initial conditions (1.2) and the conditions (2.25a) and (2.26) for $0 \leqq t < \rho$, i.e. T_j is a solution of (2.21) where A is given by (2.22) and S_j is a solution of

$$S_j'' + j^2 B S_j = 0 \tag{3.3}$$

where

$$B = a_0 + a_1 \sum_{l=1}^{\infty} l^2 S_l^2. \tag{3.4}$$

The difference

$$U_j = T_j - S_j \tag{3.5}$$

will be a solution of

$$U_j'' + j^2 A U_j = j^2 (B - A) S_j, \tag{3.6}$$

and satisfy the initial conditions

$$U_j(0) = U_j'(0) = 0. \tag{3.7}$$

The object is to show that the only solution of (3.6) satisfying (3.7) is the trivial solution. If it could be shown that U_j, or some positive definite form involving U_j, satisfied a Gronwall inequality (cf. [6]) the result would follow. However, due to the form of (3.6), it is not clear that there exists such an inequality for U_j, and thus a different approach is necessary.

It is convenient to begin by finding bounds on the solutions of (3.6). For this purpose define a function

$$E_j = (U_j')^2/j^2A + U_j^2 \geqq 0. \tag{3.8}$$

After differentiating (3.8), the differential equation (3.6) yields

$$\begin{aligned} E_j' &= -\frac{A'}{A}\left(\frac{(U_j')^2}{j^2A}\right) + 2\,\frac{(B-A)}{A}\,U_j'S_j \\ &\leqq \frac{|A'|}{A}\,E_j + 2\,\frac{|B-A|}{A}\,|U_j'|\,|S_j|, \end{aligned} \tag{3.9}$$

or equivalently

$$E_j \leqq 2\exp\left(\int_0^t \frac{|A'|}{A}\,d\tau\right)\int_0^t \frac{|B-A|}{A}\,|U_j'|\,|S_j|\,d\tau. \tag{3.10}$$

After summing over j (3.10) becomes

$$\sum_{j=1}^{\infty} E_j \leqq 2\exp\left(\int_0^t \frac{|A'|}{A}\,d\tau\right)\int_0^t I(\tau)d\tau \tag{3.11}$$

where

$$I(t) = \frac{|B-A|}{A}\sum_{j=1}^{\infty} |U_j'|\,|S_j|. \tag{3.12}$$

The function $I(t)$ may be estimated using the Schwarz inequality. Thus

$$\begin{aligned} |B-A| &\leqq a_1 \sum_{j=1}^{\infty} j^2|S_j+T_j|\,|U_j| \\ &\leqq a_1\left\{\sum_{j=1}^{\infty} j^4(S_j+T_j)^2 \sum_{j=1}^{\infty} U_j^2\right\}^{1/2}, \end{aligned} \tag{3.13}$$

and

$$\sum_{j=1}^{\infty} |U_j'|\,|S_j| \leqq \left\{\sum_{j=1}^{\infty} j^2S_j^2 \sum_{j=1}^{\infty} (U_j')^2/j^2\right\}^{1/2}, \tag{3.14}$$

so that

$$I(t) \leqq G(t)\left\{\sum_{j=1}^{\infty} \frac{(U_j')^2}{j^2A} \sum_{j=1}^{\infty} U_j^2\right\}^{1/2} \tag{3.15}$$

where

$$G(t) = a_1\left\{\sum_{j=1}^{\infty} \frac{j^2S_j^2}{A} \sum_{j=1}^{\infty} j^4(S_j + T_j)^2\right\}^{1/2}. \tag{3.16}$$

The important feature to observe about $G(t)$ is that it is bounded on any interval $0 \leqq t \leqq \rho^* < \rho$. Thus there exists a value of t, say $t = t_1$, such that

$$t \exp\left(\int_0^t \frac{|A'|}{A}\, d\tau\right) G(t) < 1 \tag{3.17}$$

for all t in the interval $0 \leqq t \leqq t_1$.

LEMMA 3.1. *$I(t) \equiv 0$ for t in the interval $0 \leqq t \leqq t_1$.*

PROOF. Assume the maximum value of $I(t)$ in the interval $0 \leqq t \leqq t_1$ occurs at $t = \eta$. The inequality (3.11) implies that

$$\sum_{j=1}^{\infty} E_j(\eta) \leqq 2\eta \exp\left(\int_0^\eta \frac{|A'|}{A}\, d\tau\right) I(\eta) \tag{3.18}$$

or (cf. (3.15))

$$\begin{aligned} \sum_{j=1}^{\infty} E_j(\eta) \leqq{} & 2\eta \\ & \cdot \exp\left(\int_0^\eta \frac{|A'|}{A}\, d\tau\right) G(\eta) \left\{\sum_{j=1}^{\infty} \frac{(U_j')^2}{j^2A} \sum_{j=1}^{\infty} U_j^2\right\}_{t=\eta}^{1/2}. \end{aligned} \tag{3.19}$$

However, if $I(\eta) \neq 0$, (3.17) and (3.19) imply that

$$\sum_{j=1}^{\infty} E_j(\eta) < 2\left\{\sum_{j=1}^{\infty} \frac{(U_j')^2}{j^2A} \sum_{j=1}^{\infty} U_j^2\right\}_{t=\eta}^{1/2} \tag{3.20}$$

or, recalling the definition of E_j (cf. (3.8)),

$$\left\{\left(\sum_{j=1}^{\infty} \frac{(U_j')^2}{j^2A}\right)^{1/2} - \left(\sum_{j=1}^{\infty} U_j^2\right)^{1/2}\right\}_{t=\eta}^{2} < 0. \tag{3.21}$$

This contradiction shows that $I(\eta) = 0$, and hence $I(t) \equiv 0$ for $0 \leqq t \leqq t_1$. Q.E.D.

The fact that $I(t) \equiv 0$ in the interval $0 \leqq t \leqq t_1$ combined with (3.11) shows that $U_j(t) \equiv 0$ and $U_j'(t) \equiv 0$ in the interval. However, this result is easily extended to the interval $0 \leqq t < \rho$. Assume there exists some value of j such that $U_j(t) \not\equiv 0$ for $0 \leqq t < \rho$. Let η_j be the greatest lower bound of points for which $U_j \neq 0$ and let $\eta = \text{g.l.b. } \eta_j \geqq t_1$. Since the functions U_j are continuously differentiable for $0 \leqq t < \rho$ it follows that $U_j(\eta) = U_j'(\eta) = 0$. Since $\eta < \rho$ the development of this section may be repeated (with $t = \eta$ as the initial point) to show that U_j and U_j' vanish in some interval to the right of $t = \eta$. Thus η could not be the greatest lower bound of points for which U_j does not vanish. This contradiction proves

THEOREM 3.1. *The system of equations* (1.1) *have at most one solution satisfying the initial conditions* (1.2) *and the conditions* (2.25a) *and* (2.26).

The solution of (1.1) which was constructed in §2 satisfies the conditions (2.25a) and (2.26). It follows that if the initial data (1.2) satisfies (2.5) and (2.6), the system (1.1) has, on the interval $0 \leqq t < t_c$, exactly one solution satisfying (2.25a) and (2.26), and this solution is the limit of solutions to the finite system (1.7). The conditions (2.25a) and (2.26) may be interpreted in terms of the partial differential equation (1.4) and its solution (1.3). The condition (2.26) is simply the condition that (1.4) remain hyperbolic and (2.25a) is related to the convergence of the Fourier series (1.3). Indeed (2.25a) is essentially the condition that the Fourier series (1.3) be twice differentiable with respect to x and t. In addition, the condition (2.5) and (2.6) are related to the differentiability of the initial conditions $W(x, 0)$ and $W_t(x, 0)$ in the partial differential equation (1.4). Thus, if the initial conditions $W(x, 0)$ and $W_t(x, 0)$ are sufficiently differentiable, the above discussion proves the existence of a solution to (1.4) of the form (1.3) in an interval $0 \leqq t < t_c$. Note also that if this solution does cease to exist for some value of $t \geqq t_c$ the cause will not be the unbounded growth of the solution (cf. (1.5)), but rather that it ceases to be sufficiently differentiable.

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