A CERTAIN TYPE OF LOCALLY COMPACT TOTALLY DISCONNECTED TOPOLOGICAL GROUPS[1]

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Let G be a locally compact totally disconnected topological group. If G contains a closed subgroup A which is isomorphic with a discrete finitely generated free abelian group, such that G/A is compact then we shall call G a totally disconnected two end group. In this note, we shall give a structure theorem of such groups.

Notations. $P(G)$ denotes the subset of G consisting of all compact elements of G; i.e. $g \in P(G)$ if and only if the closure of the subgroup generated by g is compact. $B(G)$ denotes the (characteristic) subgroup of G consisting of all the elements whose orbits under the inner automorphisms is relative compact, i.e. $g \in B(G)$ if and only if $\{fgf^{-1} | f \in G\}$ has compact closure.

LEMMA 1 (USAKOV [2]). *Let G be a topological group containing a compact open subgroup H. Let B be a periodic invariant subset of G with compact closure (periodic means that every element in B is a compact element in G). Then the closure of the subgroup generated by $\overline{B}$ is a compact normal subgroup of G.*

LEMMA 2. *If G is a totally disconnected locally compact group, then $P(G) \cap B(G)$ is a characteristic subgroup of G.*

PROOF. For $z \in G$, let $O(z) = \{gzg^{-1} | g \in G\}$. Suppose $x, y \in P(G) \cap B(G)$. Then $O(x) \cup O(y) \subset P(G) \cap B(G)$, and $O(x) \cup O(y)$ has compact closure; moreover it is an invariant subset of G. By Lemma 1 $\mathrm{cl}(O(x) \cup O(y))$ generates a subgroup whose closure is compact, a fortiori, $xy \in P(G) \cap B(G)$, and $P(G) \cap B(G)$ is a subgroup of G. It is characteristic, since $P(G)$ and $B(G)$ are characteristic subgroups of G.

THEOREM. *Suppose G is a locally compact totally disconnected topological group containing a closed discrete subgroup A such that*

(1) A is a finitely generated free abelian group,

(2) G/A is compact.

Then there exists a compact characteristic subgroup K of G such that

(1) $G/\mathrm{cl}(B(G))$ is compact,

(2) $\mathrm{cl}(B(G))/K$ is a discrete FC group; i.e. each conjugacy class in this group is a finite set.

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PROOF. It is easy to see that $A\subset B(G)\subset \mathrm{cl}(B(G))$. Thus $\mathrm{cl}(B(G))$ is compact.

Let F denote $\mathrm{cl}(B(G))$. Then $\mathrm{cl}(B(F))=F$, $A\subset F$ and F/A is compact. By Lemma 2, $B(F)\cap P(F)$ is a characteristic subgroup of F. Given any compact open subgroup L of F, $B(F)\cap L\subset B(F)\cap P(F)$, and $B(F)\cap L$ is dense in L. Hence $\mathrm{cl}(B(F)\cap P(F))$ is an open subgroup of F. Since $B(F)\cap P(F)\cap A=\{e\}$, e identity; also $P(F)\cap B(F)$ is open (hence closed) in $B(F)$, hence $\mathrm{cl}(P(F)\cap B(F))\cap B(F)$ $=P(F)\cap B(F)$. This implies $\mathrm{cl}(P(F)\cap B(F))\cap A=\{e\}$. Because F/A is compact, F is σ-compact. Then $\mathrm{cl}(P(F)\cap B(F))A$ is an open (hence closed) subgroup of F, and $\mathrm{cl}(P(F)\cap B(F))A/A$ is a closed subset of the compact space F/A and $\mathrm{cl}(P(F)\cap B(F))$ is compact. Let K $=\mathrm{cl}(P(F)\cap B(F))$. Then F/K is discrete. It is now clear that $B(F/K)$ $=F/K$, and the conjugacy class of any element in F/K is a finite set. This completes our proof.

REMARK. In fact, all we really need in the proof of the above theorem is: G contains a closed subgroup A with the properties:

(1) G/A is compact,

(2) $A\subset B(G)$, and

(3) $A\cap P(G)=\{e\}$.

REMARK. In general, G/A is compact does not imply that $\mathrm{cl}(B(G))$ $=G$. For example, group generated by two elements a, b subjected to the relation $a^2=b^2=e$.

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