# ON THE EXISTENCE OF INCOMPRESSIBLE SURFACES IN CERTAIN 3-MANIFOLDS

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If $M$ is the closure of the complement of a regular neighborhood of a nontrivial knot in $S^3$ then there exists a nonsingular torus $T$ embedded in $M$, which is incompressible (i.e. the inclusion $i: T \to M$ induces a monomorphism $i_*: \pi_1(T) \to \pi_1(M)$). If $F$ is any orientable closed incompressible surface embedded in $M$ then $\pi_1(M)$ contains $\pi_1(F)$ as a subgroup. L. Neuwirth [**3**, Question T] asks whether the converse is true: If $\pi_1(M)$ contains the group $\mathfrak{F}$ of a closed (orientable) surface of genus $g > 1$, does there exist a nonsingular closed surface $F$ of genus $g$ whose fundamental group is injected monomorphically into $\pi_1(M)$ by inclusion? As a partial answer we show that not for *every* such $\mathfrak{F} \subset \pi_1(M)$ there exists an incompressible $F \subset M$. The question remains open whether $M$ contains incompressible closed surfaces of genus $> 1$. We show that for torus knots $M$ does not contain such surfaces, by showing that $\pi_1(M)$ does not contain subgroups $\mathfrak{F}$.

1. **Isotopic surfaces.** Let $M$ be a compact 3-manifold (orientable or nonorientable). A "*surface $F$ in $M$*" always means a 2-sided embedded surface $F$ in $M$ such that $F \cap \partial M = \partial F$. $F$ is *incompressible* in $M$ iff $F \neq S^2$ and $\ker(i_*: \pi_1(F) \to \pi_1(M)) = 1$, where $i: F \to M$ is the inclusion. We say $M$ is *$P^2$-irreducible* iff $M$ is irreducible (every 2-sphere bounds a ball) and does not contain (2-sided) projective planes. $M$ is called *boundary-irreducible* iff $\partial M$ is a system of incompressible surfaces.

THEOREM 1. *Let $M$ be a $P^2$-irredubicle 3-manifold. Let $G$ be an incompressible surface in $M$ and $\mathfrak{F} \subset i_*\pi_1(G) \subset \pi_1(M)$. If there exists an incompressible surface $F \subset M$ such that $\partial F \subset \partial G \cap \partial F$ and $i_*\pi_1(F) = \mathfrak{F}$, then $F$ is isotopic to $G$.*

This follows from theorems obtained by Waldhausen [**5**]. In particular we need the following:

PROPOSITION [**5**, Proposition 5.4]. *Let $M$ be $P^2$-irreducible. Let $F$ and $G$ be incompressible surfaces in $M$, $\partial F \subset \partial F \cap \partial G$, such that $F \cap G$ consists of mutually disjoint simple closed curves (with transversal intersection at any curve which is not in $\partial F$). Let $H$ be a surface and suppose there is a map $f: H \times I \to M$ such that $f | H \times 0$ is a covering map onto $F$*

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*and* $f(\partial(H\times I)-H\times 0)\subset G$. *Then there is a surface* $\tilde{H}$ *and an embedding* $\tilde{H}\times I\to M$ *such that* $\tilde{H}\times 0=\tilde{F}\subset F$; $\mathrm{Cl}(\partial(\tilde{H}\times I)-\tilde{H}\times 0)=\tilde{G}\subset G$ *and* $\tilde{F}\cap G=\partial\tilde{F}$; *moreover if* $\tilde{G}\cap F\neq\partial\tilde{G}$ *then* $\tilde{F}$ *and* $\tilde{G}$ *are discs.*

Waldhausen proves this for orientable $M$, $F$, $G$, using his Lemmas 5.1 to 5.3 in [**5**]. In the nonorientable case 5.1 of [**5**] may be proved by looking at the orientable 2-sheeted covering of $M$ (see [**2**]). Then the proofs of Lemmas 5.2 to 5.4 in [**5**] go through in the nonorientable case as well, noting that $F$ and $G$ are 2-sided in $M$.

Proof of the theorem. Suppose $F$ exists. By small isotopic deformations, constant on $\partial M$, we may assume that $G\cap F$ consists of a system of closed curves, the number of which is minimal. We claim: There exists a surface $H$ homeomorphic to $F$ and a map $f\colon H\times I\to M$ such that $f|H\times 0$ is a homeomorphism onto $F$ and $f|(H\times 1\cup\partial H\times I)\subset G$. For, let $f|H\times 0$ be $i\colon F\to M$. Since $\mathfrak{F}\subset\pi_1(G)$ and $\partial F\subset\partial F\cap\partial G$, we can define the map on $H\times 0\cup\partial H\times I\cup H^{(1)}\times I$, where $H^{(1)}$ is the 1-skeleton of $H$, such that $f|H^{(1)}\times 1\subset G$. Since $G$ is incompressible, we can extend this map to a map from $\partial(H\times I)\to M$. Now $\pi_2(M)=0$ (by our assumption on $M$ and the projective plane theorem [**1**]; in fact it follows from the Hurewicz-isomorphism on the universal cover that $M$ is aspherical), therefore $f$ can be extended to a map $H\times I\to M$. The rest of the proof copies the proof of Corollary (5.5) in [**5**]: by the proposition, there exist pieces $\tilde{G}\subset G$ and $\tilde{F}\subset F$ which are parallel in $M$ such that $\tilde{F}\cap G=\partial\tilde{F}$. If $\tilde{G}\cap F\neq\partial\tilde{G}$ then $\tilde{F}\cup\tilde{G}$ bounds a ball, since $M$ is irreducible. This ball contains a piece $F'\subset F$. Deforming $F'$ out of this ball across $\tilde{G}$, we could make $F\cap G$ smaller, a contradiction. Hence we have $\tilde{G}\cap F=\partial\tilde{G}$. Therefore there exists an isotopic deformation of $F$ (constant on $F-\tilde{F}$) which throws $\tilde{F}$ onto $\tilde{G}$. If $\tilde{F}$ would not be all of $F$, then we could deform $\tilde{F}-\partial F\cap\tilde{F}$ out of $G$ (keeping $\partial F$ fixed) and thereby reduce the intersection number $F\cap G$. Hence $F=\tilde{F}$, $F\cap G=\partial F\subset\partial F\cap\partial G$, hence $\partial\tilde{G}=\tilde{G}\cap F\subset\partial M$ and since $G\cap\partial M=\partial G$ we have $G=\tilde{G}$.

Let $\mathfrak{F}$ be a subgroup of $\pi_1(M)$. We say $\mathfrak{F}$ *is carried by a surface* $F\subset M$ iff there exists an embedding $i\colon F\to M$ such that $i_*\pi_1(F)=\mathfrak{F}$ and $\ker i_*=1$.

Corollary. *Let* $M$ *be* $P^2$*-irreducible. Let* $G$ *be a closed incompressible surface of genus* $>1$ *in* $M$. *Then there exists a subgroup* $\mathfrak{F}\subset\pi_1(M)$ *which is not carried by a surface* $F\subset M$ *but is isomorphic to* $\pi_1(F)$. (*In fact, if* $G$ *is not a Klein bottle there exist infinitely many nonisomorphic subgroups of* $\pi_1(M)$ *having this property.*)

Proof. Let $F$ be a finite covering of $G$ such that $F$ is not homeomorphic to $G$. (Since $G\neq S^2$, $P^2$, Torus, Klein bottle, we can construct infinitely many topologically different compact $F$'s.) Then $p_*\pi_1(F)=\mathfrak{F}$ (where $p$: $F\to G$ is the covering map) is a subgroup of $\pi_1(G)$, hence of $i_*\pi_1(G)\subset\pi_1(M)$. If $\mathfrak{F}$ would be carried by $F$, then by Theorem 1, $F$ would be isotopic to $G$, a contradiction.

In particular this corollary applies to complements of nontrivial knots as mentioned in the introduction.

2. **Surfaces in 3-manifolds which groups have a center.** Let $\mathfrak{F}$ be the fundamental group of a closed surface $F$. If $F$ is orientable suppose genus $(F)>1$, if $F$ is nonorientable let genus $(F)>2$.

LEMMA. *Let $M$ be an irreducible (compact) 3-manifold with $\pi_1(M)\approx\mathfrak{F}\times\mathbf{Z}$, then $M$ is a fibre bundle over $S^1$ with fiber $F$.*

This is a special case of Stallings theorem [**4**].

THEOREM 2. *Let $M$ be a $P^2$-irreducible, boundary irreducible 3-manifold and suppose the center $\mathfrak{Z}$ of $\pi_1(M)$ is infinite. If $\partial M\neq\varnothing$, then $\pi_1(M)$ does not contain a subgroup $\mathfrak{F}$ as above.*

PROOF. Suppose there exists $\mathfrak{F}\subset\pi_1(M)$. Then, since the center of $\mathfrak{F}$ is trivial, $\mathfrak{F}\cap\mathfrak{Z}=1$. If $t\in\mathfrak{Z}$ is of infinite order, the subgroup in $\pi_1(M)$ which is generated by $\mathfrak{F}$ and $t$ is isomorphic to $\mathfrak{F}\times\mathbf{Z}(t)$. If $D(M)$ denotes the double of $M$, then since $M$ is boundary irreducible, $i_*$: $\pi_1(M)\to\pi_1(D(M))$ is a monomorphism, where $i$: $M\to D(M)$ is the inclusion (this is well known; a proof may be found, e.g., in [**4**]). Since $D(M)$ is $P^2$-irreducible and $\pi_1(D(M))$ not finite, $D(M)$ is aspherical (see the remark in the proof of Theorem 1). Therefore we can construct a map $f$: $F\times S^1\to D(M)$ which induces the embedding $\mathfrak{F}\times\mathbf{Z}\to\pi_1(M)\xrightarrow{i_*}\pi_1(D(M))$. It follows from Waldhausen's theorem [**5**, Theorem 6.1] (see [**2**] for the nonorientable case), that $f$ is homotopic to a covering map. In particular, since $F\times S^1$ is compact it follows that $\mathfrak{F}\times\mathbf{Z}$ has finite index in $\pi_1(D(M))$ and therefore in $\pi_1(M)\subset\pi_1(D(M))$. Now consider the covering $\tilde{M}$ of $M$ which is associated to $\mathfrak{F}\times\mathbf{Z}$. $\tilde{M}$ is compact. Now the universal covering of $M$ can be embedded in a ball such that the interior of this ball is contained in the embedding ([**5**, Theorem 8.1]; the proof in the nonorientable case is quite similar, since the only thing needed is the existence of a hierarchy [**2**]). Hence $\tilde{M}$ does not contain fake 3-cells, and since $\pi_2(\tilde{M})=0$ it follows that $\tilde{M}$ is irreducible. By the lemma, $\tilde{M}$ is a fiber bundle with fiber $F$, in particular $\tilde{M}$ is closed, which is absurd.

The first part of the proof gives us immediately:

PROPOSITION. *Let $M$ be a closed $P^2$-irreducible 3-manifold and sup-*

*pose the center* $\mathfrak{Z}$ *of* $\pi_1(M)$ *is infinite. If* $\pi_1(M)$ *contains a subgroup* $\mathfrak{F}$ *then* $F\times S^1$ *is a covering of* $M$.

COROLLARY (TO THEOREM 2). *The groups*

$$\begin{aligned}|\, & t_1, \cdots, t_m, g_1, \cdots, g_n, a_1, b_1, \cdots, a_p, b_p, h: \\ & t_iht_i^{-1} = h;\, g_ihg_i^{-1} = h;\, a_iha_i^{-1} = h;\, b_ihb_i^{-1} = h; \\ & g_i^{\alpha_i}h^{\beta_i} = 1, (\alpha_i, \beta_i) = 1, t_1 \cdots t_m g_1 \cdots g_n \textstyle\prod_{i=1}^{p} [a_i, b_i] = h^b, \quad b \in \mathbf{Z}\,|\end{aligned}$$

*do not contain a subgroup* $\mathfrak{F}$.

These are fundamental groups of Seifert fiber spaces. In particular the groups of torus knots $|g, h: g^\alpha h^\beta = 1|$ do not contain a subgroup $\mathfrak{F}$. Hence the complement of a torus knot does not contain closed incompressible surfaces other than Tori.

REMARK. The nonexistence of closed surfaces of genus $>1$ in irreducible orientable 3-manifolds $M$ with nonempty boundary for which $\pi_1(M)$ has nontrivial center follows immediately from Waldhausen's papers [6], [7]. In [6] Waldhausen proves that these manifolds are Seifert fiber spaces and in [7, §(10.3)] it is remarked that any incompressible surface in $M$ which is not boundary-parallel either consists of Seifert fibers (but does not contain singular fibers) or is a branched covering over the Seifert surface ("Zerlegungsfläche").

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